

The relativistic two-body potentials of constraint theory from summation of Feynman diagrams

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Abstract

The relativistic two-body potentials of constraint theory for systems composed of two spin-0 or two spin- $\frac{1}{2}$ particles are calculated, in perturbation theory, by means of the Lippmann–Schwinger type equation that relates them to the scattering amplitude. The cases of scalar and vector interactions with massless photons are considered. The two-photon exchange contributions, calculated with covariant propagators, are globally free of spurious infra-red singularities and produce at leading order $O(\alpha^4)$ effects that can be represented in three-dimensional x -space by local potentials proportional to $(\alpha/r)^2$. An approximation scheme, that adapts the eikonal approximation to the bound state problem, is devised and applied to the evaluation of leading terms of higher order diagrams. Leading contributions of n -photon exchange diagrams produce terms proportional to $(\alpha/r)^n$. The series of leading contributions are summed. The resulting potentials are functions, in the c.m. frame, of r and of the total energy. Their forms are compatible with Todorov’s minimal substitution rules proposed in the quasipotential approach.

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1 Introduction

The three-dimensional reduction of the Bethe–Salpeter equation [1, 2, 3] was achieved in the past with the techniques of the quasipotential approach [4, 5, 6, 7, 8, 9, 10, 11, 12]. The method consisted in iterating the Bethe–Salpeter equation around a three-dimensional hypersurface in relative momentum space; the result is a three-dimensional wave equation for the relative motion, where the potential is related to the scattering amplitude by means of a Lippmann-Schwinger type equation. This wave equation is not unique in form and depends among others on the three-dimensional hypersurface chosen for the projection operation, reflecting the way of eliminating the relative energy and relative time variables from the initial four-dimensional equation. A variant of this approach consisted in iterating the Bethe–Salpeter kernel around a simpler three-dimensional one [13, 14].

Quantitative applications of these three-dimensional methods concerned mainly QED and the positronium system. It turns out that the Coulomb gauge is the most convenient gauge to achieve the perturbative calculations [11, 13, 14, 15, 16]. Covariant gauges [17], or covariant propagators for scalar exchanges [18], have the tendency to produce spurious infra-red singularities that are only cancelled by contributions of higher order diagrams. This feature considerably reduces the domain of practical applicability of the Bethe–Salpeter equation in covariant perturbative calculations. Actually, this flaw can be traced back to the noncovariant nature of the three-dimensional equation around which iteration is accomplished. Also, the instantaneous approximation [19], applied to covariant propagators, does not produce the correct $O(\alpha^4)$ terms in perturbation theory, neither the correct infinite mass limit, unless an infinite number of higher order diagrams are taken into account [20].

In this respect, the variant of the quasipotential approach, developed by Todorov [10], where, to some extent, potentials may appear in local form in x -space with appropriate c.m. total energy dependences, provides a manifestly covariant framework that is free of the abovementioned diseases. In particular, it has been shown [21], for one spin- $\frac{1}{2}$ and one spin-0 particle systems, that in the two-photon exchange diagrams, calculated in the Feynman gauge, the spurious infra-red singularities cancel out and correct results are obtained for the $O(\alpha^4)$ and $O(\alpha^5 \ln \alpha^{-1})$ effects. This approach could not, however, reach sufficient generality, because of the lack in it of covariant wave equations for two-spin- $\frac{1}{2}$

particle systems.

It is the manifestly covariant formalism of constraint theory [22], applied to two-particle systems [23, 24, 25], that provided the general framework for setting up covariant wave equations with the correct number of degrees of freedom. Two-spin- $\frac{1}{2}$ fermion systems are described there by two compatible Dirac type equations [26, 27, 28]. In local approximations of the potentials, these equations, without loss of their covariance property, can be reduced to a single Pauli-Schrödinger type equation [29] which displays their practical applicability to spectroscopic problems. The connection of these equations with the Bethe–Salpeter equation is obtained with standard iteration techniques [30, 31]. (For a comparison of various three-dimensional equations see Ref. [32].)

The purpose of the present paper is to investigate the perturbation theory properties of the potentials of constraint theory wave equations. We shall not consider here radiative corrections, which actually are calculable independently from any three-dimensional formalism, but rather shall analyze the structure of multiphoton-exchange diagrams.

Our analysis is accomplished in three steps. Firstly, we study the structure of two-photon exchange diagrams and show that the corresponding potential is free of spurious infra-red logarithmic singularities and yields the correct $O(\alpha^4)$ effects; it can be represented, in three-dimensional x -space, by means of a local function, proportional to $(\alpha/r)^2$. Secondly, we devise an approximation method for the evaluation of the above diagrams that globally reproduces the same result for their leading effect. This method is a variant of the eikonal approximation [33, 34, 35], adapted to the bound state problem.

Thirdly, we apply the approximation method to the higher order diagrams. We show that the n -photon exchange diagrams yield for the corresponding potential, as a dominant contribution, an $O(\alpha^{2n})$ effect, represented by a term proportional to $(\alpha/r)^n$. We then sum the series of dominant contributions to obtain the potential in compact form, as a function of (α/r) and of the c.m. total energy.

The expression we find for the timelike component of the electromagnetic potential can naturally be completed to include the spacelike components as well. It then represents the fermionic generalization of the potential proposed by Todorov in his quasipotential approach for two-spin-0 particle systems, on the basis of minimal substitution rules [10]. The same calculations, applied to scalar photons, yield a potential that is equivalent, in the static approximation of its higher order terms, to the form proposed by Crater and

Van Alstine [27], who extended Todorov's substitution rules to scalar interactions.

The above calculations result in sizable effects mainly in strong coupling problems. One such problem is provided by the evaluation of the short-distance potential in QCD, which is very similar to the QED potential, with values of the coupling constant of the order of 0.5; furthermore, part of the radiative corrections could be taken into account by the replacement of the coupling constant by the effective coupling constant, obtained from renormalization group analysis.

Another domain of application of these potentials is provided by QED in its strong coupling regime, where lattice calculations signal a chiral phase transition for values of α of the order of 0.3-0.4 [36, 37]. Other qualitative results obtained from the Bethe–Salpeter equation with one-photon or one-gluon exchanges (such as spontaneous breakdown of chiral symmetry) might be tested with these potentials and their stability in the presence of multiphoton or multigluon exchanges checked. Also, these potentials could naturally be generalized for the treatment of other interactions, such as those corresponding to the exchanges of massive particles or to confining interactions, by the replacement there of the Coulomb potential by another appropriate potential.

The paper is organized as follows. In Sec. 2 the wave equations of constraint theory are introduced. In Sec. 3, their connection with the Bethe–Salpeter equation and the relationship of the potential with the scattering amplitude are established. In Sec. 4, the perturbative calculational method of the potential is developed. In Sec. 5, the potential corresponding to the exchanges of two scalar photons for two-spin-0 particle systems is calculated to order α^4 . In Sec. 6, similar calculations are done for scalar and vector photons for a fermion-antifermion system. In Sec. 7, the approximation method leading to the results of Secs. 5 and 6 is devised and some of its general consequences about cancellation of diagrams are shown. In Secs. 8 and 9, the dominant contributions coming from multiphoton exchanges are calculated and summed for two-boson and fermion-antifermion systems, respectively. In Sec. 10, the Bethe–Salpeter wave function is reconstituted in terms of the constraint theory wave function in the framework of the present approximations. Conclusion follows in Sec. 11. Details of the calculations of two-photon exchange diagrams are presented in Appendices A and B. A summary of the calculation of the vector potential was presented by the present authors in Ref. [38].

2 Two-body wave equations of constraint theory

A system of two-spin-0 particles, with masses m_1 and m_2 , respectively, is described by means of two wave equations that are generalizations of the Klein–Gordon equation [23, 24, 25, 27, 28]:

$$(p_1^2 - m_1^2 - V) \Psi(x_1, x_2) = 0 , \quad (2.1a)$$

$$(p_2^2 - m_2^2 - V) \Psi(x_1, x_2) = 0 , \quad (2.1b)$$

where V is a Poincaré invariant function or operator of the variables of the system. It is possible to choose different potentials in the two equations, but by means of canonical (or wave function) transformations one can bring the corresponding equations to the above configuration [39].

We use the following definitions for the total and relative variables:

$$\begin{aligned} P &= p_1 + p_2 , \quad p = \frac{1}{2}(p_1 - p_2) , \quad M = m_1 + m_2 , \\ X &= \frac{1}{2}(x_1 + x_2) , \quad x = x_1 - x_2 . \end{aligned} \quad (2.2)$$

For states that are eigenstates of the total momentum P we define transverse and longitudinal components of four-vectors with respect to P :

$$\begin{aligned} q_\mu^T &= q_\mu - \frac{q \cdot P}{P^2} P_\mu , \quad q_\mu^L = (q \cdot \hat{P}) \hat{P}_\mu , \quad \hat{P}_\mu = \frac{P_\mu}{\sqrt{P^2}} , \quad q_L = q \cdot \hat{P} , \\ P_L &= \sqrt{P^2} , \quad q_L \Big|_{c.m.} = q_0 , \quad q^{T^2} \Big|_{c.m.} = -\mathbf{q}^2 , \quad r = \sqrt{-x^{T^2}} . \end{aligned} \quad (2.3)$$

This decomposition is manifestly covariant. In the c.m. frame the transverse components reduce to the three space components, while the longitudinal component reduces to the time component of the corresponding four-vector.

The compatibility condition of the two wave equations (2.1a)-(2.1b) imposes the following constraints on the wave function and the potential:

$$[(p_1^2 - p_2^2) - (m_1^2 - m_2^2)] \Psi = 0 , \quad (2.4)$$

$$\left[p_1^2 - p_2^2, V \right] \Psi = 0 . \quad (2.5)$$

The solution of Eq. (2.4) is (for eigenfunctions of the total momentum P):

$$\Psi = e^{-iP \cdot X} e^{-i(m_1^2 - m_2^2)x_L/(2P_L)} \psi(x^T) , \quad (2.6)$$

while Eq. (2.5) tells us that V does not depend on the relative longitudinal coordinate x_L :

$$V = V(x^T, P_L, p^T) . \quad (2.7)$$

Thus, Eqs. (2.4)-(2.5) allow us to eliminate the longitudinal relative coordinate and momentum. Equation (2.4) can be rewritten in a slightly different form by defining the constraint $C(p)$:

$$C(p) \equiv 2P_L p_L - (m_1^2 - m_2^2) \approx 0 . \quad (2.8)$$

It yields for the individual longitudinal momenta the expressions:

$$p_{1L} = \frac{P_L}{2} + \frac{(m_1^2 - m_2^2)}{2P_L} , \quad p_{2L} = \frac{P_L}{2} - \frac{(m_1^2 - m_2^2)}{2P_L} . \quad (2.9)$$

The eigenvalue equation satisfied by the internal wave function ψ is:

$$\left[\frac{P^2}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + p^{T2} \right] \psi(x^T) = 0 , \quad (2.10)$$

which is a three-dimensional Klein–Gordon–Schrödinger type equation. Also notice that when constraint C (2.8) is utilized, the two individual Klein–Gordon operators become equal:

$$H_0 \equiv (p_1^2 - m_1^2) \Big|_C = (p_2^2 - m_2^2) \Big|_C = \frac{P^2}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4P^2} + p^{T2} . \quad (2.11)$$

In general the potential V [Eq. (2.7)] is a Poincaré invariant integral operator in x^T , but in simplified cases it can be represented as a function of x^T with a general dependence on P_L , and an eventual quadratic dependence on p^T .

The scalar product for the wave functions satisfying Eqs. (2.1a)-(2.1b) can be obtained either from the construction of tensor currents satisfying two independent conservation laws [28] with respect to x_1 and x_2 , respectively, or directly from the Green's function of Eqs. (2.1a)-(2.1b) or (2.10) [8, 11]. The result for the norm is (in the c.m. frame):

$$\int d^3\mathbf{x} \, \psi^* 4P^2 \left[\frac{\partial}{\partial P^2} (H_0 - V) \right] \psi = 2P_0 , \quad (2.12)$$

where H_0 is the Klein–Gordon operator (2.11).

For a system of two-spin- $\frac{1}{2}$ particles, composed of one fermion with mass m_1 and one antifermion with mass m_2 , the constraint theory wave equations can be written in the form [28]:

$$(\gamma_1 \cdot p_1 - m_1) \tilde{\Psi} = (-\gamma_2 \cdot p_2 + m_2) \tilde{V} \tilde{\Psi} , \quad (2.13a)$$

$$(-\gamma_2 \cdot p_2 - m_2) \tilde{\Psi} = (\gamma_1 \cdot p_1 + m_1) \tilde{V} \tilde{\Psi} , \quad (2.13b)$$

where \tilde{V} is a Poincaré invariant potential. Here, $\tilde{\Psi}$ is a sixteen-component spinor wave function of rank two and is represented as a 4×4 matrix:

$$\tilde{\Psi} = \tilde{\Psi}_{\alpha_1 \alpha_2}(x_1, x_2) \quad (\alpha_1, \alpha_2 = 1, \dots, 4) , \quad (2.14)$$

where α_1 (α_2) is the spinor index of particle 1 (2). γ_1 is the Dirac matrix γ acting in the subspace of the spinor of particle 1 (index α_1); it acts on $\tilde{\Psi}$ from the left. γ_2 is the Dirac matrix acting in the subspace of the spinor of particle 2 (index α_2); it acts on $\tilde{\Psi}$ from the right; this is also the case of products of γ_2 matrices, which act on $\tilde{\Psi}$ from the right in the reverse order:

$$\begin{aligned} \gamma_{1\mu} \tilde{\Psi} &\equiv (\gamma_\mu)_{\alpha_1 \beta_1} \tilde{\Psi}_{\beta_1 \alpha_2} , & \gamma_{2\mu} \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2} (\gamma_\mu)_{\beta_2 \alpha_2} , \\ \gamma_{2\mu} \gamma_{2\nu} \tilde{\Psi} &\equiv \tilde{\Psi}_{\alpha_1 \beta_2} (\gamma_\nu \gamma_\mu)_{\beta_2 \alpha_2} , & \sigma_{a\alpha\beta} &\equiv \frac{1}{2i} [\gamma_{a\alpha}, \gamma_{a\beta}] \quad (a = 1, 2) . \end{aligned} \quad (2.15)$$

The compatibility condition of Eqs. (2.13a)-(2.13b) yields the same constraints as in Eqs. (2.4) and (2.5) (with Ψ and V replaced by $\tilde{\Psi}$ and \tilde{V} , respectively). In particular, one has:

$$\tilde{V} = \tilde{V}(x^T, P_L, p^T, \gamma_1, \gamma_2) . \quad (2.16)$$

Using the operator H_0 [Eq. (2.11)], Eqs. (2.13a)-(2.13b) can also be rewritten as a single equation, provided the constraint (2.8) is used:

$$\left[\frac{1}{i^2} (\gamma_1 \cdot p_1 - m_1) (-\gamma_2 \cdot p_2 - m_2) H_0^{-1} \right]_{C(p)} \tilde{\Psi} \equiv \left[S_1(p_1) S_2(-p_2) H_0 \right]_{C(p)}^{-1} \tilde{\Psi} = -\tilde{V} \tilde{\Psi} , \quad (2.17)$$

where S_1 and S_2 are the propagators of fermions 1 and 2, respectively.

The norm for the internal wave function $\tilde{\psi}$ takes the form (in the c.m. frame):

$$\int d^3 \mathbf{x} \, Tr \left\{ \tilde{\psi}^\dagger \left[1 - \tilde{V}^\dagger \tilde{V} + 4 \gamma_{10} \gamma_{20} P_0^2 \frac{\partial \tilde{V}}{\partial P^2} \right] \tilde{\psi} \right\} = 2P_0 , \quad (2.18)$$

with \tilde{V} satisfying the hermiticity condition

$$\tilde{V}^\dagger = \gamma_{10} \gamma_{20} \tilde{V} \gamma_{10} \gamma_{20} . \quad (2.19)$$

Equation (2.18) shows that, for energy independent potentials (in the c.m. frame) the norm of $\tilde{\psi}$ is not positive definite. In order to ensure positivity, it is sufficient that the potential \tilde{V} satisfy the inequality

$$\frac{1}{4} Tr(\tilde{V}^\dagger \tilde{V}) < 1 . \quad (2.20)$$

In this case one is allowed to make the wave function transformation

$$\tilde{\Psi} = \left[1 - \tilde{V}^\dagger \tilde{V}\right]^{-1/2} \Psi \quad (2.21)$$

and to reach a representation where the norm for c.m. energy independent potentials is the free norm.

In this respect, the parametrization suggested by Crater and Van Alstine [40], for local potentials that commute with $\gamma_{1L}\gamma_{2L}$ (hence $\tilde{V}^\dagger = \tilde{V}$),

$$\tilde{V} = \tanh V, \quad (2.22)$$

satisfies condition (2.20) and allows one to bring the equations satisfied by Ψ [Eq. (2.21)] into forms analogous to the Dirac equation, where each particle appears as placed in the external potential created by the other particle, the latter potential having the same tensor nature as potential V of Eq. (2.22).

We shall henceforth adopt parametrization (2.22) for potential \tilde{V} and, according to Eq. (2.21), shall introduce the wave function transformation

$$\tilde{\Psi} = (\cosh V) \Psi. \quad (2.23)$$

[For potentials that do not commute with $\gamma_{1L}\gamma_{2L}$, the generalizations of Eqs. (2.22) and (2.23) and of the corresponding norm are presented in Ref. [41].] The norm of the internal part of the new wave function Ψ then becomes (in the c.m. frame):

$$\int d^3\mathbf{x} \, Tr \left\{ \psi^\dagger \left[1 + 4\gamma_{10}\gamma_{20}P_0^2 \frac{\partial V}{\partial P^2} \right] \psi \right\} = 2P_0. \quad (2.24)$$

[The relationship between Ψ and ψ is the same as in Eq. (2.6).]

In order to obtain the final eigenvalue equation of ψ , one has to decompose the latter along 2×2 components. We choose, for this decomposition, the basis of the matrices 1, γ_L , γ_5 and $\gamma_L\gamma_5$:

$$\psi = \psi_1 + \gamma_L\psi_2 + \gamma_5\psi_3 + \gamma_L\gamma_5\psi_4. \quad (2.25)$$

When potential V is a function of x^T (i.e., not an integral operator) it is possible, for some general classes of potential, to eliminate through the wave equations (2.13a)-(2.13b) three of these components in terms of a fourth one. We choose ψ_3 for the remaining component, which is also a surviving component in the nonrelativistic limit. For a general

combination of scalar, pseudoscalar and vector potentials, the final eigenvalue equation is of the Pauli–Schrödinger type, the radial momentum operator appearing in it through the Laplace operator. For this class of potential, V is decomposed along the γ matrices as:

$$V = V_1 + \gamma_{15}\gamma_{25}V_3 + \gamma_1^\mu\gamma_2^\nu(g_{\mu\nu}^{LL}V_2 + g_{\mu\nu}^{TT}U_4 + \frac{x_\mu^T x_\nu^T}{x^{T2}}T_4) , \quad (2.26)$$

where V_1 is the scalar, V_3 the pseudoscalar and V_2 the timelike vector potential; U_4 and T_4 are the spacelike vector potentials. [The numbering of the indices and the notations are related to the particular projectors to which these potentials are attached [29].]

After making a change of function of the type

$$\psi_3 = A\phi_3 , \quad (2.27)$$

the expression of A being given in Ref. [29], one finds the following Pauli–Schrödinger type equation (in the c.m. frame):

$$\left\{ e^{4(U_4 + T_4)} \left[\frac{P^2}{4} e^{4V_2} - \frac{M^2}{4} e^{4V_1} - \frac{(m_1^2 - m_2^2)^2}{4M^2} e^{-4V_1} + \frac{(m_1^2 - m_2^2)^2}{4P^2} e^{-4V_2} \right] \right. \\ \left. - \mathbf{p}^2 - \mathbf{L}^2 \frac{1}{\mathbf{x}^2} (e^{4T_4} - 1) + \dots \right\} \phi_3 = 0 , \quad (2.28)$$

where the dots stand for the various spin dependent or \hbar^2 -dependent terms and \mathbf{L} is the orbital angular momentum operator. [The complete form of the equation can be found in Ref. [29].] We recognize that the scalar potential V_1 acts as a modification of the total mass M of the fermions through the change $M \rightarrow Me^{2V_1}$ while $(m_1^2 - m_2^2)$ is kept fixed. The timelike vector potential V_2 acts as a modification of the c.m. total energy P_L through the change $P_L \rightarrow P_L e^{2V_2}$, while $(p_{1L}^2 - p_{2L}^2) = (m_1^2 - m_2^2)$ is kept fixed. The nonmodification of $(m_1^2 - m_2^2)$ is simply due to the constraint (2.8). The spacelike potential U_4 changes the orbital angular momentum operator from \mathbf{L} to $\mathbf{L}e^{-2U_4}$ and the combination $U_4 + T_4$ of the spacelike potentials changes the radial momentum operator from p_r to $p_r e^{-2(U_4 + T_4)}$ (in the classical limit). The pseudoscalar potential appears only in spin dependent and \hbar^2 -dependent terms. All these effects are what one expects from an external field interpretation of the potentials in the Dirac and Klein–Gordon equations and therefore provide additional justification for the parametrization (2.22) and the decomposition of V along the form (2.26).

Wave equations can also be written down for a system made of one spin- $\frac{1}{2}$ and one spin-0 particles [28]. We shall not, however, consider such a system in the present work.

3 Connection with the Bethe–Salpeter equation and the scattering amplitude

In order to find the connection of the wave equations (2.1a)-(2.1b) and (2.13a)-(2.13b) with the Bethe–Salpeter equation, it is natural to project, with an appropriate weight factor, four-dimensional quantities such as Green’s functions, scattering amplitudes and wave functions on the constraint hypersurface (2.8) and to iterate the corresponding integral equations around that hypersurface. There is an arbitrariness in the choice of the weight factor, but it is limited by several additional conditions, such as correct one-body limit, absence of spurious singularities, correct $O(\alpha^4)$ effects, interchange symmetry between particles 1 and 2, hermiticity of the potential, etc.. We shall first present the general method of approach and then shall discuss the question of the choice of the weight factor.

Let $G(P, p, p')$ be the four-point Green’s function (for bosons or fermions) with total momentum P and relative momenta p' and p for the ingoing and outgoing particles, respectively [Eqs. (2.2)], and from which the total four-momentum conservation factor $(2\pi)^4\delta^4(P - P')$ has been amputated. It satisfies the integral equation:

$$G = G_0 + G_0 K G , \quad (3.1)$$

where G_0 is the free two-particle propagator,

$$G_0 = G_1 G_2 \quad (3.2)$$

[G_1 and G_2 are the propagators of particles 1 and 2, respectively], and K is the Bethe–Salpeter kernel [1, 2, 3]. The off-mass shell scattering amplitude T (from which the total four-momentum conservation factor has been amputated) is related to K by the relation:

$$T = K + K G_0 T . \quad (3.3)$$

In the vicinity of a bound state pole, G behaves as

$$\lim_{s \rightarrow s_B} G(P, p, p') = i\eta \frac{\phi(P, p) \bar{\phi}(P, p')}{s - s_B + i\epsilon} , \quad s = P^2 , \quad \eta = \begin{cases} +1 & \text{for bosons,} \\ -1 & \text{for fermions,} \end{cases} \quad (3.4)$$

where ϕ is the internal part of the Bethe–Salpeter wave function and $\bar{\phi}$ is defined from hermitian conjugation and antichronological product.

We next define the left-projected Green's function on the hypersurface (2.8):

$$\tilde{G}(p, p') = -2i\pi\delta(C(p))G_{0C}^{-1}G(p, p') , \quad (3.5)$$

where G_{0C}^{-1} is the weight factor to be determined below. [When not necessary, total four-momentum will not be explicitly written in the arguments.] Notice that \tilde{G} is still four-dimensional with respect to its right argument.

It is now possible to iterate G around \tilde{G} and obtain the integral equation for \tilde{G} . We shall use the following definitions and abbreviated notations:

$$\begin{aligned} \delta_C &= \delta(C(p)) , & g &= G_0 + 2i\pi\delta_C G_{0C}^{-1}G_0 , \\ \tilde{g} &= +2i\pi\delta_C G_{0C}^{-1}G_0 , & \tilde{g}_0 &= G_{0C}^{-1}G_0 \Big|_{C(p)} . \end{aligned} \quad (3.6)$$

Using in the right-hand side of Eq. (3.5) the integral equation for G , one obtains for the difference $G - \tilde{G}$:

$$G - \tilde{G} = g(1 + KG) , \quad (3.7)$$

which yields:

$$G = (1 - gK)^{-1}(\tilde{G} + g) . \quad (3.8)$$

One then successively uses Eqs. (3.1) and (3.8) in the right-hand side of Eq. (3.5) to obtain:

$$\tilde{G} = -\tilde{g} \left[(1 - Kg)^{-1} + K(1 - gK)^{-1}\tilde{G} \right] . \quad (3.9)$$

The second term in the right-hand side of this equation can also be expressed in terms of the scattering amplitude with the use of Eq. (3.3):

$$\tilde{G} = -\tilde{g} \left[(1 - Kg)^{-1} + T(1 - \tilde{g}T)^{-1}\tilde{G} \right] . \quad (3.10)$$

We define the constraint theory wave function ψ with the same projection as in Eq. (3.5):

$$\psi = -2i\pi\delta_C G_{0C}^{-1}\phi , \quad (3.11)$$

which leads to the following behavior of \tilde{G} in the neighborhood of a bound state pole:

$$\lim_{s \rightarrow s_B} \tilde{G}(p, p') = i\eta \frac{\psi(p)\bar{\phi}(p')}{s - s_B + i\epsilon} \quad (3.12)$$

[η is defined in Eq. (3.4)], which, in turn, yields the wave equation satisfied by ψ :

$$\psi = -\tilde{g}T(1 - \tilde{g}T)^{-1}\psi . \quad (3.13)$$

We shall also define the reduced wave function $\tilde{\psi}$ and amplitude \tilde{T} as:

$$\psi = 2\pi 2P_L \delta_C \tilde{\psi} , \quad (3.14a)$$

$$\tilde{T}(p, p') = \frac{i}{2P_L} T(p, p') \Big|_{C(p), C(p')} . \quad (3.14b)$$

The wave equation satisfied by $\tilde{\psi}$ takes the form:

$$\tilde{g}_0^{-1} \tilde{\psi} = -\tilde{T}(1 - \tilde{g}_0 \tilde{T})^{-1} \tilde{\psi} , \quad (3.15)$$

where now the integrations between \tilde{T} , \tilde{g}_0 and $\tilde{\psi}$ are three-dimensional and concern the transverse variables [Eqs. (2.3)], after the constraint (2.8) has been used for the longitudinal momenta.

It is also possible to reconstitute the Green's function G and the Bethe–Salpeter wave function ϕ from the knowledge of the constraint theory Green's function \tilde{G} and wave function ψ . To this end, we consider Eq. (3.7), in the right-hand side of which we replace G by its expression (3.8), and use Eq. (3.9) for \tilde{G} ; we find:

$$G = G_0 + G_0 T (1 - \tilde{g} T)^{-1} (\tilde{G} + g) , \quad (3.16)$$

$$\phi = G_0 T (1 - \tilde{g} T)^{-1} \psi . \quad (3.17)$$

Notice that in the left argument of the first T in the above equations the constraint (2.8) is not used. We also dispose for G of Eq. (3.8), which yields for ϕ the relation

$$\phi = (1 - gK)^{-1} \psi , \quad (3.18)$$

but these equations are more singular than Eqs. (3.16) and (3.17) above, and should only be used inside integrals.

It is natural at this stage to define the potential \tilde{V} (also denoted by V for the bosonic case) by the Lippmann–Schwinger type relation:

$$\tilde{V} = \tilde{T}(1 - \tilde{g}_0 \tilde{T})^{-1} . \quad (3.19)$$

Equation (3.15) then becomes:

$$\tilde{g}_0^{-1} \tilde{\psi} = -\tilde{V} \tilde{\psi} . \quad (3.20)$$

This equation has the same structure as the constraint theory wave equations (2.1a)–(2.1b) and (2.17). Comparison of \tilde{g}_0^{-1} with the corresponding kinematic operators then fixes the expression of G_{0C}^{-1} [Eqs. (3.5) and (3.6)]:

$$G_{0C}^{-1} \Big|_{C(p)} = H_0 . \quad (3.21)$$

[H_0 is defined in Eq. (2.11).]

Let us now examine the hermiticity property of potential \tilde{V} , which should partly be guaranteed by the on-mass shell elastic unitarity property of the scattering amplitude, through relation (3.19). After a little algebra [10], one finds for the bosonic case the following condition on \tilde{g}_0 :

$$\tilde{g}_0 - \tilde{g}_0^* = 2i\pi\delta(b_0^2(s) + p^{T2})\theta(s - (m_1 + m_2)^2), \quad (3.22)$$

where we have defined

$$\begin{aligned} b_0^2(s) &= \frac{1}{4}(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2) \\ &= \frac{s}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4s}. \end{aligned} \quad (3.23)$$

The argument of the δ -function above is nothing but the operator H_0 defined in Eq. (2.11), which means that solution (3.21) for G_{0C}^{-1} automatically satisfies the hermiticity-unitarity condition (3.22) in the physical region of the amplitude. [G_{0C} should be defined with the prescription $(H_0 + i\epsilon)^{-1}$.]

For the fermionic case, the hermiticity-unitarity condition reads:

$$\begin{aligned} \tilde{g}_0 - \gamma_{1L}\gamma_{2L}\tilde{g}_0^\dagger\gamma_{1L}\gamma_{2L} &= 2i\pi\delta(b_0^2(s) + p^{T2})\theta(s - (m_1 + m_2)^2) \\ &\quad \times (\gamma_1 \cdot p_1 + m_1)(-\gamma_2 \cdot p_2 + m_2), \end{aligned} \quad (3.24)$$

which is also satisfied by solution (3.21).

Other solutions to Eqs. (3.22) or (3.24), different from that given in Eq. (3.21), can be constructed as well. In principle, different choices of \tilde{g}_0 should not affect the physical predictions of the theory, but rather would amount to organizing in different ways the resolution of the bound state problem and the summation of the perturbation series, putting emphasis on specific approximations. In the present work, we stick to the choice (3.21), which has the two main advantages of simplicity and of the fact that it reduces in x -space to a second-order differential operator. It is only in this case that we can device local approximation schemes for potential \tilde{V} and remain in a framework analogous to that of nonrelativistic quantum mechanics. Choice (3.21) has also the advantages of hermiticity, interchange symmetry between particles 1 and 2 and correct one-body limit.

We shall, however, slightly generalize solution (3.21) by allowing in it the presence of a finite multiplicative renormalization constant. Such a generalization is necessary in the

off-mass shell formalism we are using to avoid the presence of spurious $O(\alpha^3)$ -terms. It amounts to requiring that the $1/r$ -terms of the potential result solely from the contribution of the one-photon exchange diagram. For this purpose we define

$$\tilde{g}'_0 = \gamma \tilde{g}_0 , \quad (3.25)$$

where \tilde{g}_0 corresponds to solution (3.21) and γ is a constant, expressible as a series of the coupling constant. Equation (3.20) then becomes, after replacement of \tilde{g}_0 by \tilde{g}'_0 :

$$\tilde{g}_0^{-1} \tilde{\psi} = -\gamma \tilde{V} \tilde{\psi} , \quad (3.26)$$

with

$$\tilde{V} = \tilde{T}(1 - \gamma \tilde{g}_0 \tilde{T})^{-1} . \quad (3.27)$$

The presence of the factor γ in Eqs. (3.26) and (3.27) leaves enough freedom for accomplishing the desired cancellation. For the simplicity of notation, we shall no longer write explicitly the factor γ , but when necessary shall mention its presence.

The integral equation satisfied by the Green's function \tilde{G} also allows one to construct with standard techniques [8, 11] the scalar product of states.

4 Perturbation theory calculation of the potential

Our main task in the remaining part of this work is the calculation, in perturbation theory and within a definite approximation scheme, of potential \tilde{V} from formula (3.19). The perturbation series of \tilde{V} contains, in addition to the usual Feynman diagrams, other types of diagram, arising from the presence of the constraint factor \tilde{g}_0 ; we shall call these diagrams “constraint diagrams”; they are obtained from the usual box-ladder type diagrams by the replacement of (at least) one pair of fermion and antifermion (or boson) propagators by the constraint propagator \tilde{g} [Eq. (3.6)]. The role of these diagrams will be to cancel the spurious singularities (referred to the potential) arising from the calculation of the amplitude and to yield a potential that has most of the required physical properties.

It was already shown in the case of fermions [29] that, when one of the particles becomes infinitely massive, the contributions of the constraint diagrams cancel those of the ladder and crossed ladder diagrams, leaving only the contribution of the one-photon exchange diagram. Thus the wave equation of the remaining particle reduces in this limit to the Dirac equation in the presence of the static potential created by the infinitely massive particle.

In the present work we limit ourselves to the evaluation of the ladder and crossed ladder diagrams in their leading order, neglecting radiative corrections. In this approximation the scattering amplitude \tilde{T} [Eq. (3.14b)] can be decomposed as:

$$\tilde{T} = \sum_{n=1}^{\infty} \tilde{T}^{(n)} , \quad (4.1)$$

where $\tilde{T}^{(n)}$ is the partial amplitude corresponding to n exchanged photons.

Iteration of Eq. (3.19) yields for the potential the expansion:

$$\tilde{V} \equiv \sum_{n=1}^{\infty} \tilde{V}^{(n)} = \tilde{T} \sum_{p=0}^{\infty} (\tilde{g}_0 \tilde{T})^p , \quad (4.2)$$

where $\tilde{V}^{(n)}$ is that part of the potential which comes from n exchanged photons. The first two terms of \tilde{V} are:

$$\tilde{V}^{(1)} = \tilde{T}^{(1)} , \quad (4.3)$$

$$\tilde{V}^{(2)} = \tilde{T}^{(2)} + \tilde{T}^{(1)} \tilde{g}_0 \tilde{T}^{(1)} . \quad (4.4)$$

The corresponding diagrams are represented, for the case of fermions, in Figs. 1 and 2,

respectively. [The ingoing particles have now momenta p_1 and p_2 and the outgoing ones p'_1 and p'_2 . The constraint diagram is represented by a box with a cross in it.]

Our evaluation of the importance of various terms hinges on the infra-red counting rules of the bound state problem in QED. Thus, if α is the dimensionless coupling constant squared, r [Eq. (2.3)] counts as $O(\alpha^{-1})$, the transverse momentum p^T as $O(\alpha)$, the transferred momentum squared t as $O(\alpha^2)$, etc.. Furthermore, we shall evaluate terms with respect to x -space; therefore, expressions computed in momentum space will undergo three-dimensional Fourier transformations and will receive additional $O(\alpha^3)$ contributions; thus α/t will be counted as $O(\alpha^2)$, $\alpha^2/\sqrt{(-t)}$ as $O(\alpha^4)$, etc..

The one-photon exchange contribution, $\tilde{V}^{(1)}$, is of order α^2 and, in three-dimensional x -space, is proportional to α/r . As we shall show in Secs. 5 and 6, the two-photon exchange contributions, corresponding to $\tilde{V}^{(2)}$ [Eq. (4.4)], yield as a dominant effect an $O(\alpha^4)$ term, proportional, in x -space to α^2/r^2 . It is therefore expected, generalizing these results, that the n -photon exchange contributions yield as a dominant effect an $O(\alpha^{2n})$ term, proportional to $(\alpha/r)^n$. We shall show, in Secs. 7,8 and 9, within an approximation scheme, that this is indeed the case. Therefore, the sum of leading terms of each $\tilde{V}^{(n)}$ in Eq. (4.2) will result in a local function of α/r .

With the above counting rules, there are still nonleading terms, in $\tilde{V}^{(n)}$ ($n > 1$), say, that might compete with leading terms of $\tilde{V}^{(n')}$ with $n' > n$. Such terms have, however, a different structure than a simple power of α/r ; they are essentially momentum dependent and cannot be grouped within a function of α/r ; therefore, they will not be considered in the summation process, although, in a quantitative perturbation theory calculation their effects, which begin at $O(\alpha^6)$, should be taken into account, like those of the radiative corrections.

To have a consistency check of our summation procedure, we make, in the following sections, parallel calculations of the scalar potential (with scalar massless photons) in the bosonic and fermionic cases, where the Feynman diagrams have rather different structures. One however expects that the spin independent classical part of the potential in the fermionic case, when brought in an appropriate representation, should coincide with the bosonic potential. This coincidence does occur and provides additional justification for our summation rules.

5 Two scalar photon exchange contribution in bosonic system

We consider in this section a system of two spin-0 particles, with masses m_1 and m_2 , respectively, interacting with scalar (massless) photons. The corresponding lagrangian density is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu A\partial^\mu A + \partial_\mu\phi_1^\dagger\partial^\mu\phi_1 - \phi_1^\dagger(m_1^2 + 2m_1gA + g^2A^2)\phi_1 + (1 \rightarrow 2) . \quad (5.1)$$

We have kept analogy with QED and introduced the interactions with the photon with the substitutions $m_1 \rightarrow m_1 + gA$ and $m_2 \rightarrow m_2 + gA$. We define:

$$\alpha = \frac{g^2}{4\pi}. \quad (5.2)$$

Furthermore, we ignore other types of interaction like ϕ^4 or A^4 which arise from renormalization. We assume that the corresponding renormalized coupling constants have been put equal to zero; more generally, these coupling constants are of order $g^4 \sim \alpha^2$ and the corresponding Feynman diagrams yield contact potentials that are of order α^5 and hence can be neglected in comparison to $O(\alpha^4)$ effects.

Our definitions of the propagators are:

$$G_1(p_1) = \frac{i}{p_1^2 - m_1^2 + i\epsilon}, \quad G_2(p_2) = \frac{i}{p_2^2 - m_2^2 + i\epsilon}, \quad D(k) = \frac{i}{k^2 + i\epsilon}. \quad (5.3)$$

In the process $(1) + (2) \rightarrow (1') + (2')$ we designate the corresponding momenta by p_1, p_2, p'_1 and p'_2 , respectively, and introduce the usual momentum variables:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p'_1 + p'_2)^2 = P^2 = P_L^2, \\ q &= p_1 - p'_1 = p'_2 - p_2, \quad t = q^2, \quad u = (p_1 - p'_2)^2. \end{aligned} \quad (5.4)$$

When constraint (2.8) is imposed on the external particles, one obtains:

$$p_{1L} = p'_{1L}, \quad p_{2L} = p'_{2L}, \quad q_L = 0, \quad q^2 = q^{T2}, \quad (5.5)$$

with p_{1L} and p_{2L} given by Eqs. (2.9).

The Feynman diagrams will be calculated with the external particles considered off the mass shell, with longitudinal momenta fixed by the bound state mass P_L , through relations (2.9), while the transverse momenta squared p^{T2} and p'^{T2} have, according to our

counting rules, orders of magnitude of α^2 . The off-mass shell deviations are represented by the quantities

$$\lambda^2 = m_1^2 - p_1^2 = m_2^2 - p_2^2, \quad \lambda'^2 = m_1^2 - p_1'^2 = m_2^2 - p_2'^2, \quad (5.6)$$

which are positive and are of order α^2 . The mean values of p^{T2} and p'^{T2} in the bound state are equal and hence we have a similar equality for λ^2 and λ'^2 :

$$\langle p^{T2} \rangle = \langle p'^{T2} \rangle, \quad \langle \lambda^2 \rangle = \langle \lambda'^2 \rangle. \quad (5.7)$$

To the accuracy of the present calculations, where we are evaluating only leading effects of a set of Feynman diagrams with a fixed number of photons, we shall not distinguish between p^{T2} and p'^{T2} , and between λ^2 and λ'^2 , and shall use the equalities (5.7) without the mean values.

We also introduce the off-mass shell analogue of the quantity $b_0^2(s)$ [Eq. (3.23)]:

$$b^2(s) = \frac{s}{4} - \frac{1}{2}(p_1^2 + p_2^2) + \frac{(p_1^2 - p_2^2)^2}{4s} = -p^{T2}, \quad (5.8)$$

the second equality arising from the use of Eqs. (2.4), (2.8), and (2.9) and of the definitions $p_1^2 = p_{1L}^2 + p^{T2}$, $p_2^2 = p_{2L}^2 + p^{T2}$. One also easily shows the relations:

$$b_0^2 = p_{1L}^2 - m_1^2 = p_{2L}^2 - m_2^2 = -\lambda^2 + b^2. \quad (5.9)$$

Notice that $b^2 > 0$, while $b_0^2 < 0$. Keeping only the leading order terms in α , one also has the mean value equality:

$$\langle \lambda^2 \rangle = 2 \langle b^2 \rangle, \quad (5.10)$$

which in turn implies the leading order equality:

$$(p^T + p'^T)^2 = -2(\lambda^2 + \frac{t}{2}). \quad (5.11)$$

We first calculate the one-photon exchange contribution (Fig. 1). Because of the constraint conditions [Eqs. (5.5)], the momentum carried by the photon is transverse, $q = (q_L = 0, q^T)$ [$q^{T2} = -\mathbf{q}^2$ in the c.m. frame]; the passage to x -space is done with the three-dimensional Fourier transformation with respect to q^T . Taking into account the definition of \tilde{T} [Eq. (3.14b)], one finds for the potential $V^{(1)}$ [Eq. (4.3)] (the potential in the bosonic case is not tilded):

$$V^{(1)} = \frac{2m_1 m_2 g^2}{P_L t}, \quad (5.12)$$

or, in x -space:

$$V^{(1)} = -\frac{2m_1m_2}{P_L} \frac{\alpha}{r}. \quad (5.13)$$

We next calculate the two-photon exchange contributions. In addition to the diagrams of Fig. 2, one also has the two diagrams of Fig. 3. We first introduce the notations for the various integrals we shall meet during the calculations. They are similar to those introduced by Brown and Feynman [42] and Redhead [43] (but our volume element d^4k does not contain a $(2\pi)^{-2}$ factor.) We use the abbreviated notations:

$$\begin{aligned} (0) &= k^2 + i\epsilon, & (3) &= (q - k)^2 + i\epsilon, \\ (1) &= (p_1 - k)^2 - m_1^2 + i\epsilon, & (2) &= (p_2 + k)^2 - m_2^2 + i\epsilon, \\ (-1') &= (-p'_1 - k)^2 - m_1^2 + i\epsilon, & (-2') &= (-p'_2 + k)^2 - m_2^2 + i\epsilon; \end{aligned} \quad (5.14)$$

$$\begin{aligned} J_{,\mu,\mu\nu} &= \int d^4k \frac{(1, k_\mu, k_\mu k_\nu)}{(0)(3)(1)(2)}, \\ F_{,\mu} &= \int d^4k \frac{(1, k_\mu)}{(3)(1)(2)}, & H_{,\mu} &= \int d^4k \frac{(1, k_\mu)}{(0)(1)(2)}, \\ G_{,\mu}^{(1)} &= \int d^4k \frac{(1, k_\mu)}{(0)(3)(1)}, & G_{,\mu}^{(2)} &= \int d^4k \frac{(1, k_\mu)}{(0)(3)(2)}. \end{aligned} \quad (5.15)$$

For the crossed diagrams, where (2) is replaced by $(-2')$, we use the abbreviated notations:

$$J_{,\mu,\mu\nu}(1, -2') = \int d^4k \frac{(1, k_\mu, k_\mu k_\nu)}{(0)(3)(1)(-2')}, \quad \text{etc.} \quad (5.16)$$

For the integrals resulting from constraint diagrams and according to the definitions (3.6), (3.21), (3.14b) and (3.19), we have to suppress one of the boson propagator denominator, (1) or (2), and replace it by $2i\pi\delta(C)/(2P_L)$, the operation being symmetric with respect to the interchange $(1) \leftrightarrow (2)$:

$$\begin{aligned} J_{,\mu,\mu\nu}^C &= \frac{2i\pi}{2P_L} \int d^4k \delta(k_L) \frac{(1, k_\mu^T, k_\mu^T k_\nu^T)}{(0)(3)(1)}, \\ F_{,\mu}^C &= \frac{2i\pi}{2P_L} \int d^4k \delta(k_L) \frac{(1, k_\mu^T)}{(3)(1)}, & H_{,\mu}^C &= \frac{2i\pi}{2P_L} \int d^4k \delta(k_L) \frac{(1, k_\mu^T)}{(0)(1)}, \\ G_{,\mu}^C &= \frac{2i\pi}{2P_L} \int d^4k \delta(k_L) \frac{(1, k_\mu^T)}{(0)(3)}. \end{aligned} \quad (5.17)$$

Details of the calculations of the above integrals are presented in Appendices A and B (see also Ref. [21]). For the integrals of the diagrams of Fig. 2 we find:

$$\begin{aligned}
J = & \frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma[\gamma\lambda^2 - (1-\gamma)t]} \left(\frac{\lambda^2}{\gamma s} - \frac{b^2}{s} \right)^{-\frac{1}{2}} \\
& + \frac{2i\pi^2}{st} \left[-\frac{P_L^2}{2p_{1L}p_{2L}} \ln\left(-\frac{(\lambda^2)^2}{st}\right) + \frac{P_L}{p_{1L}} \ln\left(\frac{p_{1L}}{P_L}\right) + \frac{P_L}{p_{2L}} \ln\left(\frac{p_{2L}}{P_L}\right) + \frac{P_L^2}{p_{1L}p_{2L}} \right] \\
& + O(\alpha^3 \ln \alpha^{-1}) ,
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
J(1, -2') = & -\frac{2i\pi^2}{st} \left[-\frac{P_L^2}{2p_{1L}p_{2L}} \ln\left(-\frac{(\lambda^2)^2}{st}\right) + \frac{P_L^2}{p_{1L}p_{2L}} \right. \\
& \left. - \frac{P_L^2}{p_{1L}(p_{1L} - p_{2L})} \ln\left(\frac{p_{1L}}{P_L}\right) + \frac{P_L^2}{p_{2L}(p_{1L} - p_{2L})} \ln\left(\frac{p_{2L}}{P_L}\right) \right] + O(\alpha^3 \ln \alpha^{-1}) ,
\end{aligned} \tag{5.19}$$

$$J^C = -\frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma[\gamma\lambda^2 - (1-\gamma)t]} \left(\frac{\lambda^2}{\gamma s} - \frac{b^2}{s} \right)^{-\frac{1}{2}} . \tag{5.20}$$

By taking the sum of J , $J(1, -2')$ and J^C , one finds that J^C cancels the dominant singularity in J , while the next-to-leading singularity of J cancels the leading singularity of $J(1, -2')$. One obtains:

$$J + J(1, -2') + J^C = \frac{2i\pi^2}{t} \frac{1}{(p_{1L}^2 - p_{2L}^2)} \ln\left(\frac{p_{1L}^2}{p_{2L}^2}\right) + O(\alpha^3 \ln \alpha^{-1}) . \tag{5.21}$$

Taking into account the coupling constants and the other multiplicative factors, the potential $V^{(2)}$ resulting from the above contribution becomes:

$$V^{(2)J} = -\frac{m_1^2 m_2^2 g^4}{P_L t} \frac{1}{\pi^2 (m_1^2 - m_2^2)} \ln\left(\frac{m_1^2}{m_2^2}\right) + O(\alpha^5 \ln \alpha^{-1}) . \tag{5.22}$$

This is a spurious $O(\alpha^3)$ -term, since the potential is known, from other semirelativistic calculations, not to possess such a term. It is a two-body effect and cannot be removed by one-particle renormalizations. The origin of this term lies in the off-mass shell extrapolation of the Lippmann–Schwinger formula (3.19) with the use of Eq. (3.21). We actually also calculated the contribution of the three diagrams of Fig. 2 on the mass shell, giving to the photon a small mass. We found that the sum of the three terms of Eq. (5.21) vanishes (up to $O(\alpha^3 \ln \alpha^{-1})$) and hence the term (5.22) is absent from the potential. If we admit

that at leading orders on-mass shell and off-mass shell calculations should coincide as far as the potential is concerned, we find another justification for removing the above term by an appropriate renormalization of the Lippmann–Schwinger formula. This prescription amounts to requiring that the $1/r$ -terms in the potential arise solely from the one-photon exchange diagram. We described the corresponding procedure through Eqs. (3.25) to (3.27). Choosing to this order

$$\gamma = 1 + \gamma_1, \quad \gamma_1 = \frac{g^2}{2\pi^2} \frac{m_1 m_2}{(m_1^2 - m_2^2)} \ln\left(\frac{m_1^2}{m_2^2}\right), \quad (5.23)$$

one can cancel, with the aid of $V^{(1)}$ [Eq. (5.12)], the term $V^{(2)J}$ [Eq. (5.22)]:

$$V^{(2)J} + \gamma_1 V^{(1)} = 0. \quad (5.24)$$

Therefore, the net effect of the three diagrams of Fig. 2 can be considered as zero to order α^4 in the potential $V^{(2)}$.

We next evaluate the contributions of the diagrams of Fig. 3. The corresponding integrals are:

$$G^{(a)} = -\frac{i\pi^4}{2p_{aL}\sqrt{-t}} + O(\alpha^3 \ln \alpha^{-1}) \quad (a = 1, 2), \quad (5.25)$$

and their contribution to the potential $V^{(2)}$ is [we recall that there is a symmetry factor 2 at the four-line vertex]:

$$V^{(2)G} = \frac{1}{8P_L} \left(\frac{m_1^2}{p_{1L}} + \frac{m_2^2}{p_{2L}} \right) \frac{g^4}{\sqrt{-t}} + O(\alpha^5 \ln \alpha^{-1}). \quad (5.26)$$

In x -space, the potential $V^{(2)}$ takes the form:

$$V^{(2)} = \frac{1}{P_L} \left(\frac{m_1^2}{p_{1L}} + \frac{m_2^2}{p_{2L}} \right) \frac{\alpha^2}{r^2}, \quad (5.27)$$

and, together with $V^{(1)}$ [Eq. (5.13)], the potential becomes to order α^4 :

$$V = V^{(1)} + V^{(2)} = -\frac{2m_1 m_2}{P_L} \frac{\alpha}{r} + \frac{1}{P_L} \left(\frac{m_1^2}{p_{1L}} + \frac{m_2^2}{p_{2L}} \right) \frac{\alpha^2}{r^2}. \quad (5.28)$$

In summary, the constraint diagram has cancelled the dominant spurious logarithmic singularity of the direct diagram, while the remaining subdominant logarithmic singularities have been cancelled among the direct and crossed diagrams. Among the two types of diagram we met in Figs. 2 and 3, it is only the diagrams of the type of Fig. 3 (the “seagull” diagrams) that have contributed to the $O(\alpha^4)$ effects.

Similar calculations can also be repeated with scalar QED (vector photons). We did such calculations in an arbitrary covariant linear gauge (characterized by a parameter ξ). The vector nature of the interaction renders the calculations more complicated. Furthermore, the constraint diagram has an ultra-violet divergence and necessitates an appropriate subtraction. The final qualitative results remain, however, the same as above: the spurious singularities disappear and the leading term is of order α^4 , with the correct coefficient as expected from scalar QED spectroscopy [10, 16] and the Breit hamiltonian for bosons. We shall not present the details of these calculations in this article and shall be content with the relatively simple case of scalar interactions for the bosonic case.

6 Scalar and vector interactions with fermions

We turn in this section to the calculation of the potentials resulting from one- and two-scalar or vector photon exchanges between a fermion with mass m_1 and an antifermion with mass m_2 . The interaction lagrangian densities are

$$\mathcal{L}_{SI} = g\bar{\psi}_1 A\psi_1 + (1 \rightarrow 2) , \quad \mathcal{L}_{VI} = e\psi_1\gamma_\mu A^\mu\psi_1 + (1 \rightarrow 2) , \quad (6.1)$$

respectively, with the common notation for the coupling constant squared, Eq. (5.2) and

$$\alpha = \frac{e^2}{4\pi} . \quad (6.2)$$

The fermion propagators are:

$$S_1(p_1) = \frac{i}{\gamma_1 \cdot p_1 - m_1^2 + i\epsilon} , \quad S_2(-p_2) = \frac{i}{-\gamma_2 \cdot p_2 - m_2^2 + i\epsilon} , \quad (6.3)$$

respectively; the scalar photon propagator is given in Eqs. (5.3), while the vector photon propagator is, in the linear gauge characterized by a parameter ξ :

$$D_{\mu\nu}(k) = -(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2}) \frac{i}{k^2 + i\epsilon} . \quad (6.4)$$

The constraint factor \tilde{g}_0 [Eqs. (3.6), (3.2), (3.21) and (2.11)] is:

$$\tilde{g}_0 = S_1(p_1)S_2(-p_2)H_0 \Big|_{C(p)} . \quad (6.5)$$

The potential $\tilde{V}^{(1)}$ that results from one scalar photon exchange is:

$$\tilde{V}_S^{(1)} = \frac{g^2}{2P_L t} \quad (6.6)$$

[$t = q^{T2}$, when constraint C is used on the external particles, Eqs. (5.5)], which becomes in x -space:

$$\tilde{V}_S^{(1)} = -\frac{\alpha}{2P_L r} . \quad (6.7)$$

For the vector interaction one obtains for $\tilde{V}^{(1)}$:

$$\tilde{V}_V^{(1)} = -\frac{e^2}{2P_L} \gamma_1^\mu \gamma_2^\nu (g_{\mu\nu} - \xi \frac{q_\mu^T q_\nu^T}{t}) \frac{1}{t} , \quad (6.8)$$

which becomes in x -space:

$$\tilde{V}_V^{(1)} = \frac{\alpha}{2P_L r} \gamma_1^\mu \gamma_2^\nu (g_{\mu\nu} - g_{\mu\nu}^{TT} \frac{\xi}{2} + \frac{x_\mu^T x_\nu^T}{x^{T2}} \frac{\xi}{2}) . \quad (6.9)$$

Upon comparing this expression with Eq. (2.26) [notice that the difference between \tilde{V} and V [Eq. (2.22)] shows up only at third formal order in α] one identifies the vector potentials V_2 , U_4 and T_4 in lowest order:

$$V_2 = \frac{\alpha}{2P_L r}, \quad U_4 = (1 - \frac{\xi}{2}) \frac{\alpha}{2P_L r}, \quad T_4 = \frac{\xi}{2} \frac{\alpha}{2P_L r}. \quad (6.10)$$

In the Feynman gauge ($\xi = 0$) one observes the simple relations:

$$V_2 = U_4, \quad T_4 = 0. \quad (6.11)$$

The difference between the timelike vector potential V_2 and the spacelike vector potentials U_4 and T_4 is that, at each given formal order in α , the latter, because of the multiplicative γ^T -matrices, contribute to terms that are $O(\alpha^2)$ times less important than the contribution of the former. This feature makes the computation of the potentials U_4 and T_4 rather tricky, since they contribute only to nonleading terms, while V_2 contributes to the leading ones. In the present work we shall only calculate V_2 . However, if one makes the assumption that relations (6.11) obtained in lowest order in the Feynman gauge remain valid in higher orders, the knowledge of V_2 will allow the reconstitution of the whole vector potential in this gauge.

For the calculation of $\tilde{V}^{(2)}$ we need to calculate the contributions of the three diagrams of Fig. 2. To this end, we transform the fermion propagators by bringing the γ -matrices in the numerators and giving the denominators a bosonic form. Notice that the passage from the direct to the crossed diagram is obtained by the replacement $p_2 + k \rightarrow p'_2 - k$, which, in the bosonic denominators is equivalent to $(-p'_2 + k)$. We then obtain for the scalar potential $\tilde{V}_S^{(2)}$ the following decomposition in terms of the integrals (5.15)-(5.17):

$$\begin{aligned} \tilde{V}_S^{(2)} \left(\frac{i}{2P_L} \frac{g^4}{(2\pi)^4} \right)^{-1} &= (\gamma_1 \cdot p_1 + m_1)(-\gamma_2 \cdot p_2 + m_2)(J + J(1, -2') + J^C) \\ &\quad - (\gamma_1 \cdot p_1 + m_1)\gamma_2^\mu (J_\mu - J_\mu(1, -2') + J_\mu^C) \\ &\quad - (-\gamma_2 \cdot p_2 + m_2)\gamma_1^\mu (J_\mu + J_\mu(1, -2') + J_\mu^C) \\ &\quad + \gamma_1^\mu \gamma_2^\nu (J_{\mu\nu} - J_{\mu\nu}(1, -2') + J_{\mu\nu}^C) \\ &\quad - \left[(\gamma_1 \cdot p_1 + m_1)J(1, -2') - \gamma_1 \cdot J(1, -2') \right] \gamma_2 \cdot q. \end{aligned} \quad (6.12)$$

The transverse matrices γ^T are counted as being of order α , since they connect nondominant components of the wave function to the dominant one. We summarize here the

leading contributions in x -space of the integrals of Eq. (6.12) (see Appendices A and B):

$$\begin{aligned}
J + J(1, -2') + J^C &= O(\alpha^3 \ln \alpha^{-1}) , \\
J_L &= \frac{1}{2P_L}(G^{(1)} - G^{(2)}) , \quad J_L(1, -2') = -\frac{1}{2P_L}(G^{(1)} + G^{(2)}) , \\
\gamma_2^T \cdot (J^T - J^T(1, -2') + J^C) &= O(\alpha^3 \ln \alpha^{-1}) , \\
\gamma_1^T \cdot (J^T + J^T(1, -2') + J^C) &= O(\alpha^3 \ln \alpha^{-1}) , \\
\gamma_1^\mu \gamma_2^\nu (J_{\mu\nu} - J_{\mu\nu}(1, -2') + J_{\mu\nu}^C) &= O(\alpha^3 \ln \alpha^{-1}) , \\
\left[(\gamma_1 \cdot p_1 + m_1) J(1, -2') - \gamma_1 \cdot J(1, -2') \right] \gamma_2 \cdot q &= O(\alpha^3 \ln \alpha^{-1}) .
\end{aligned} \tag{6.13}$$

The first relation above is understood in the sense of Eq. (5.24). The expressions of $G^{(1)}$ and $G^{(2)}$ are given in Eqs. (5.25).

After replacing the matrices γ_{1L} and γ_{2L} by their eigenvalues $+1$ and -1 , respectively, with respect to the dominant component of the wave function and making the approximations $m_1 \approx p_{1L}$, $m_2 \approx p_{2L}$, one finally obtains:

$$\tilde{V}_S^{(2)} = \frac{g^4}{16P_L^2} \frac{1}{\sqrt{-t}} , \tag{6.14}$$

which, in x -space, is equivalent to:

$$\tilde{V}_S^{(2)} = \frac{\alpha^2}{2P_L^2 r^2} . \tag{6.15}$$

The scalar potential is then, up to two-photon exchanges:

$$\tilde{V}_S = \tilde{V}_S^{(1)} + \tilde{V}_S^{(2)} = -\frac{\alpha}{2P_L r} + \frac{\alpha^2}{2P_L^2 r^2} . \tag{6.16}$$

For the vector potential one has to add the γ -matrices at the vertices. For the direct and constraint diagrams one obtains expressions having the following structure: $\gamma_1 \cdot \gamma_2$ [] $\gamma_1 \cdot \gamma_2$, where the quantity [] is the corresponding expression obtained in the scalar interaction case. For the crossed diagram the structure is: $\gamma_{1\alpha} \gamma_2^\beta$ [] $\gamma_{1\beta} \gamma_2^\alpha$, with [] representing the corresponding expression of the scalar case. The interchange of the matrices γ_2^β and γ_2^α brings in new terms, which, however, turn out to be of order $\alpha^3 \ln \alpha^{-1}$ compared to the right-hand side of Eq. (6.12). Therefore, to order α^4 in the potential, the contribution of the three diagrams can be obtained from the right-hand side of Eq.

(6.12) with the global substitution $\gamma_1 \cdot \gamma_2 \rightarrow \gamma_1 \cdot \gamma_2$ and the cancellations met in the scalar case operate again. At leading order, one finds for the timelike vector potential:

$$\tilde{V}_V^{(2)} = -\gamma_{1L}\gamma_{2L} \frac{e^4}{16P_L^2} \frac{1}{\sqrt{-t}} , \quad (6.17)$$

or, in x -space:

$$\tilde{V}_V^{(2)} = -\gamma_{1L}\gamma_{2L} \frac{\alpha^2}{2P_L^2 r^2} . \quad (6.18)$$

The timelike vector potential is then, up to two-photon exchanges:

$$\tilde{V}_V = \tilde{V}_V^{(1)} + \tilde{V}_V^{(2)} = \gamma_{1L}\gamma_{2L} \left(\frac{\alpha}{2P_L r} - \frac{\alpha^2}{2P_L^2 r^2} \right) . \quad (6.19)$$

It can be shown [29] that when the two-photon exchange contribution (6.18) is added to the complete one-photon exchange contribution (6.10)-(6.11) and the nonrelativistic limit of the wave equations (2.13a)-(2.13b) is taken to order $1/c^2$, and after a wave function transformation that is equivalent to the passage from the Feynman gauge to the Coulomb gauge, one obtains the Breit hamiltonian [44] which is known to yield the correct bound state spectrum to order α^4 . This is also an indication that no other $O(\alpha^4)$ -terms should arise from higher order diagrams. (Verification of correct $O(\alpha^4)$ -terms with Todorov's electromagnetic potential was presented in Ref. [45].)

Result (6.15) was also shown [29] to contain the correct retardation effects, calculated in Refs. [46, 47] by different methods.

In summary, the same type of cancellations as for the bosonic case has occurred for the fermions, despite the fact that the tensor structure of the Feynman diagrams is more complicated. The terms that contribute to the leading $[O(\alpha^4)]$ effects are those coming from the longitudinal components of the vector integrals J_μ . In these integrals, the linear contribution of the longitudinal component of the photon momentum of the numerator cancels the dominant contribution of one of the bosonic denominators of the fermion propagators and transforms the whole integral into a G -type integral (5.15). All happens as if the fermions had effective interactions of the type of Fig. 3.

7 Cancellations with constraint diagrams

The fact that the two-photon exchange diagrams globally produce in leading order (α^4), both for the bosonic and fermionic cases, local potentials with the correct coefficients, as compared to the bound state spectrum analysis to order $1/c^2$, suggests that such a result might also hold with n -photon exchange ($n > 2$) diagrams; in this case, we should expect to have a local potential proportional to $(\alpha/r)^n$. It is, however, not possible to show such a result in a rigorous way, because of the complexity of the n -photon exchange diagrams. For this reason, it is necessary to devise an approximation scheme that takes into account the properties of the bound state we are considering, as well as the cancellation mechanisms present in the two-photon exchange cases.

The kinematic domain of the bound state is characterized by the fact that the transverse momenta of the external particles are damped, by an $O(\alpha)$ -factor, with respect to the longitudinal momenta and the masses. Also, the factor $1/r^2$ of the two-photon exchange potentials represents in momentum space the three-dimensional convolution (up to multiplicative coefficients) of the two photon propagators, in which the longitudinal momentum squared, k_L^2 , has been neglected; this means that the entire q^2 -dependence of the potential (at leading order) comes from the photons and not from the fermions. Finally, the expressions of the potentials we obtained are independent of the off-mass shell condition imposed on the bosons and the fermions (i.e., do not depend on $p_1^2 - m_1^2$ or $p_2^2 - m_2^2$).

It is then natural to assume that, in quantities that are expected to be globally free of infra-red singularities, the above properties also reflect themselves in the internal boson or fermion propagators. We shall therefore consider the approximation that consists in linearizing with respect to the internal photon momenta the inverse boson propagators and in neglecting in them the momentum transfer, as well as off-mass shell terms. One then obtains for the boson propagators the expressions:

$$\begin{aligned} G_1(p_1 - k) &\simeq \frac{i}{-2p_1 \cdot k + i\epsilon} , & G_2(p_2 + k) &\simeq \frac{i}{2p_2 \cdot k + i\epsilon} , \\ G_1(-p'_1 - k) &\simeq \frac{i}{2p_1 \cdot k + i\epsilon} , & G_2(-p'_2 + k) &\simeq \frac{i}{-2p_2 \cdot k + i\epsilon} . \end{aligned} \quad (7.1)$$

This approximation can be considered as a variant of the eikonal approximation [33, 26, 35] adapted to the bound state problem. Here, one expects for the components of k

the orders of magnitude

$$|k^T|, |k_L| = O(\alpha) . \quad (7.2)$$

[Actually, $|k_L|$ will come out of the order of $O(\alpha^2)$.] If the propagators (7.1) were multiplied by regular terms, we could immediately neglect in them $p^T.k^T$, but in many cases the multiplicative terms are singular and such an approximation would be misleading before the isolation of the singularities. Finally, because of the on-mass shell treatment of the boson or fermion propagators in approximations (7.1), the photon should be given a small mass to prevent infra-red divergences from occurring at intermediary stages.

It can be checked by direct calculation that approximations (7.1) yield the correct results (5.27), (6.15) and (6.18). This can also be checked from the results that we shall obtain for the general case of n -photon exchanges.

We shall show in this section that all types of diagram that have constraint diagram counterparts are globally cancelled by the latter in leading order of our approximations. The main formula we use is a generalization of an identity already used in the eikonal approximation (cf. Ref. [26], Appendix, and Ref. [20], pp. 116-117). Let $(c_1, c_2, \dots, c_{n+1})$ be a set of $(n+1)$ numbers; it can be divided into $(n+1)$ independent subsets of n numbers, where in each subset one of the c_i 's ($i = 1, \dots, n+1$) is missing. We have the identity:

$$\sum_{i=1}^{n+1} \sum_{perm} \left[(c'_1)^{-1} (c'_1 + c'_2)^{-1} \cdots (c'_1 + c'_2 + \cdots + c'_n)^{-1} \right]_{c_i} = \left(\sum_{i=1}^{n+1} c_i \right) / \left(\prod_{j=1}^{n+1} c_j \right) , \quad (7.3)$$

where $(c'_1, c'_2, \dots, c'_n)$ is a permutation of the subset of n numbers $(c_1, c_2, \dots, c_{i-1}, c_{i+1}, \dots, c_{n+1})$.

Let us consider in the bosonic case an $(n+1)$ -photon exchange diagram of the type of Fig. 4, which generalizes those of Fig. 2. We denote by k_1, \dots, k_{n+1} the momenta carried by the photons; for definiteness we number them according to their departure point on the line of boson 1, from the left to the right. To take into account momentum conservation, the factor $(2\pi)^4 \delta^4(\sum_{i=1}^{n+1} k_i - q)$ must be included in the corresponding integral. Since $q_L = 0$, we have $\sum_{i=1}^{n+1} k_{iL} = 0$; but according to the approximations (7.1) we may also neglect q^T in the boson propagators; hence, we have there the approximation

$$\sum_{i=1}^{n+1} k_i = 0 . \quad (7.4)$$

The total number of the diagrams considered above is $(n+1)!$. In a given diagram we have n propagators of boson 1 and n propagators of boson 2; only n photon momenta

appear on each of the boson lines (but not the same in general). The $(n+1)!$ diagrams can be divided into $(n+1)$ sets, where in each set we have $n!$ permutations of boson 2 propagators containing the same set of n photon momenta. With approximations (7.1), the sum of all these propagators has the structure of the left-hand side of Eq. (7.3), with $c_i = (2p_2 \cdot k_i + i\epsilon)$; hence, we obtain the factor

$$I_{2,n+1} = \frac{\sum_{i=1}^{n+1} (2p_2 \cdot k_i + i\epsilon)}{\prod_{j=1}^{n+1} (2p_2 \cdot k_j + i\epsilon)} . \quad (7.5)$$

Using Eq. (7.4), we can rewrite $I_{2,n+1}$ in the form:

$$\begin{aligned} I_{2,n+1} &= \frac{\sum_{i=1}^n (2p_2 \cdot k_i + i\epsilon)}{\prod_{j=1}^n (2p_2 \cdot k_j + i\epsilon)} \left(\frac{1}{2p_2 \cdot k_{n+1} + i\epsilon} + \frac{1}{\sum_{i=1}^n 2p_2 \cdot k_i + i\epsilon} \right) \\ &= I_{2,n} \left(\frac{1}{-\sum_{i=1}^n 2p_2 \cdot k_i + i\epsilon} + \frac{1}{\sum_{i=1}^n 2p_2 \cdot k_i + i\epsilon} \right) = -2i\pi \delta\left(\sum_{i=1}^n 2p_2 \cdot k_i\right) I_{2,n} . \end{aligned} \quad (7.6)$$

Repetition of the above operation on $I_{2,n}$, $I_{2,n-1}$, etc., yields the final result:

$$I_{2,n+1} = (-2i\pi)^n \delta(2p_2 \cdot k_1) \delta(2p_2 \cdot (k_1 + k_2)) \cdots \delta(2p_2 \cdot \sum_{i=1}^n k_i) . \quad (7.7)$$

The δ -functions can then be used, upon integrations on the k_{iL} 's, in the boson 1 propagators. Denoting by $J^{(n+1)}$ the total contribution of the above diagrams (without the coupling constants and other multiplicative coefficients), we obtain:

$$\begin{aligned} J^{(n+1)} &= \left(\frac{-i}{2P_L} \right)^n \int \left[\prod_{i=1}^{n+1} \frac{d^3 k_i^T}{(2\pi)^3} \frac{1}{(k_i^{T2} - \mu^2 + i\epsilon)} \right] (2\pi)^3 \delta^3\left(\sum_{i=1}^{n+1} k_i^T - q^T\right) \\ &\quad \times \frac{1}{(-2p^T \cdot k_1^T + i\epsilon)} \frac{1}{(-2p^T \cdot (k_1^T + k_2^T) + i\epsilon)} \cdots \frac{1}{(-2p^T \cdot \sum_{j=1}^n k_j^T + i\epsilon)} , \end{aligned} \quad (7.8)$$

where μ is a small mass given to the photon, to avoid infra-red divergence. Also, the δ -functions (7.7) yield for the k_{iL}^2 -terms orders of magnitude of $O(\alpha^4)$, which accounts for their omission in front of $O(\alpha^2)$ -terms in approximations (7.1) and in photon propagators.

On the other hand, we should also take into account the contributions of constraint diagrams that are associated with the diagrams considered above. From the general

formula (4.2) we obtain for the potential corresponding to $n + 1$ photon exchanges the expansion:

$$\tilde{V}^{(n+1)} = \tilde{T}^{(n+1)} + \sum_{p=1}^n \sum_{r_1+\dots+r_{p+1}=n+1} \tilde{T}^{(r_1)} \tilde{g}_0 \tilde{T}^{(r_2)} \tilde{g}_0 \dots \tilde{T}^{(r_p)} \tilde{g}_0 \tilde{T}^{(r_{p+1})}, \quad n \geq 1, \quad r_i \geq 1, \quad (7.9)$$

where, in the generic term of the sum, the constraint factor \tilde{g}_0 appears p times. A typical diagram, where $p = 2$, is shown in Fig. 5. The constraint diagrams corresponding to the diagrams of the type of Fig. 4 are those for which the amplitudes $\tilde{T}^{(r_i)}$ in Eq. (7.9) are also of the same type, with r_i exchanged photons. Now the longitudinal momentum transfer between the external particles and a constraint factor or between two successive constraint factors is zero; furthermore, according to approximations (7.1), we can also neglect transverse momentum transfers in the bosonic propagators between external particles and constraint factors and between two successive constraint factors. Hence, for the partial amplitudes $\tilde{T}^{(r_i)}$ ($r_i \geq 2$) in Eq. (7.9) we obtain integrals $J^{(r_i)}$ that have the same structure as in Eq. (7.8), with n replaced by $r_i - 1$, except for the factor $(2\pi)^3 \delta^3$ which is relative to the whole diagram. Next, the constraint factor \tilde{g}_0 provides the coefficient $(2i\pi)/(2P_L)$ multiplied by the boson propagator contribution $1/(-2p^T \cdot \sum_{j=1}^{r_1+r_2+\dots+r_i} k_j^T + i\epsilon)$, and so forth. Therefore, the constraint diagrams yield globally the same expressions as $J^{(n+1)}$ [Eq. (7.8)], except for a sign factor equal to $(-1)^p$ if \tilde{g}_0 occurs p times in the expansion (7.9). Moreover, the p factors \tilde{g}_0 may appear in $\binom{n}{p}$ different configurations. The total contribution of diagrams of the type of Fig. 4 and of their associated constraint diagrams is then proportional to

$$J^{(n+1)} \left[1 + \sum_{p=1}^n (-1)^p \binom{n}{p} \right] = (1 - 1)^{n+1} = 0, \quad n \geq 1. \quad (7.10)$$

This result generalizes the one obtained in the two-photon exchange case.

With diagrams involving more than two-photon exchanges, we can meet configurations that are mixtures of diagrams of the types of Fig. 4 and of generalizations of Fig. 3. In this case the permutational procedure on a given line should be applied to those parts of the diagrams for which the momentum configuration on the other line is unchanged. Furthermore, to show the cancellation mechanism by constraint diagrams, not all propagators need be permuted; it is sufficient to focus attention on those propagators that do not correspond to the basis of a seagull diagram (of the type of Fig. 3), since it is only these that may have constraint diagram counterparts. A simple example illustrates this

situation. Let us consider the six diagrams of Fig. 6. We apply on line 2 the permutational method for the first two diagrams [(a) and (b)]. The result is a δ -function which yields the constraint diagram (a) of Fig. 7 with an opposite sign and hence a mutual cancellation occurs. Similarly, graphs (c) and (d) of Fig. 6 are cancelled by the constraint graph (b) of Fig. 7.

The application of the permutations on line 2 in diagrams (e) and (f) of Fig. 6 yields also a δ -function, but the resulting expression does not have a constraint diagram counterpart. It is proportional to:

$$\delta(p_2.k_2) \frac{1}{-p_1.k_1 + i\epsilon} \frac{1}{-p_1.(k_1 + k_2) + i\epsilon} . \quad (7.11)$$

After an integration with respect to k_{2L} this expression becomes proportional to:

$$\frac{1}{p^T.k_2^T} \left(\frac{1}{-p_{1L}k_{1L} - p^T.k_1^T + i\epsilon} - \frac{1}{-p_{1L}k_{1L} - p^T.k_1^T - P_L p^T.k_2^T/p_{2L} + i\epsilon} \right) . \quad (7.12)$$

Finally, by making in the second term the changes of variable

$$k_{1L} \rightarrow k'_{1L} = k_{1L} + P_L \frac{p^T.k_2^T}{p_{1L}p_{2L}} , \quad k_{3L} \rightarrow k'_{3L} = k_{3L} - P_L \frac{p^T.k_2^T}{p_{1L}p_{2L}} , \quad (7.13)$$

the whole integral becomes proportional to:

$$\frac{1}{p^T.k_2^T} \frac{1}{(-p_{1L}k_{1L} - p^T.k_1^T + i\epsilon)} \left[D(k_{1L}^2 + k_1^{T2})D(k_{3L}^2 + k_3^{T2}) - D(k_{1L}'^2 + k_1^{T2})D(k_{3L}'^2 + k_3^{T2}) \right] . \quad (7.14)$$

With the use of assumptions (7.2), the difference of the two terms inside the brackets yields a non-leading term that can be neglected. Therefore, the six diagrams of Fig. 6 are cancelled, at leading order, by the two constraint diagrams of Fig. 7.

This result can be generalized to more complicated diagrams. Use of the permutational procedure on a given line produces a certain number of δ -functions, but only a smaller number of them survives at leading order; these are precisely those which appear in the constraint diagrams and are cancelled by them.

We thus end up with the conclusion that the only diagrams that survive (at leading order) for the calculation of the potential in the bosonic case are those which do not have constraint diagram counterparts. These are generalizations of the diagrams of Fig. 3, some examples of which are presented in Figs. 8 and 9.

The above study can also be applied to the fermionic case. For the fermion propagators we use the approximations:

$$\begin{aligned} S_1(p_1 - k_1) &\simeq \frac{i}{-2p_1.k_1 + i\epsilon} [(\gamma_{1L}p_{1L} + m_1) - \gamma_{1L}k_{1L}] , \\ S_2(-(p_2 + k_2)) &\simeq \frac{i}{2p_2.k_2 + i\epsilon} [(-\gamma_{2L}p_{2L} + m_2) - \gamma_{2L}k_{2L}] . \end{aligned} \quad (7.15)$$

In the numerators we have neglected the transverse momenta p^T and k_i^T (but not in the denominators); in products of two fermion propagators we neglect in the numerator the quadratic terms k_L^2 (of the same k_L). It can be checked that approximations (7.15) yield the correct results (6.15) and (6.18) of the two-photon exchange case.

With fermions, we have only diagrams of the type of Fig. 4, but the presence of the k_{iL} 's in the numerators of their propagators gives rise to a situation that is very similar to that of the bosons. The products of $(n - 1)$ pairs of fermion propagators (for n -photon diagrams) can be decomposed into three types of terms. The first type of term does not contain any k_{iL} in the numerator. It behaves as in the bosonic case with diagrams of the type of Fig. 4 and leads to an expression that is proportional to $J^{(n)}$ [Eq. (7.8)] and is cancelled by the corresponding constraint diagram contributions. In the second type of terms some of the k_{iL} 's (but not all) appear in the numerator. The analysis of these terms is similar to that of the mixtures of diagrams of the types of Fig. 3 and Fig. 4 we considered in the bosonic case (but with the additional possibility of having more than two photons at the effective seagull vertices) and their contributions are again cancelled by corresponding contributions of constraint diagrams. In the third type of terms products of $(n - 1)$ independent combinations of the k_{iL} 's appear in the numerator. These do not have constraint diagram counterparts, since there $k_{iL} = 0$ ($i = 1, \dots, n$). Therefore, these terms are the only surviving parts of the above diagrams for the calculation of the potential.

8 The potential in the bosonic case

We calculate in this section the scalar potential for two-boson systems. According to the analysis of Sec. 7 the diagrams that contribute in higher orders are those of the types of Figs. 8 and 9. The diagrams of Fig. 8 correspond to an odd number of exchanged photons, while those of Fig. 9 to an even number of exchanged photons.

We begin by considering diagrams of the type of Fig. 8. Let $(2n + 1)$, with $n \geq 1$, be the number of exchanged photons. The total number of such diagrams is $(n + 1)!(n + 1)!$. We denote by $k_1, k_2, \dots, k_{2n+1}$ the momenta carried by the photons, and for definiteness we number them starting from the photon that leaves the boson line 1, and then follow continuously the other photons. A global momentum conservation factor, $(2\pi)^4 \delta^4(\sum_{i=1}^{2n+1} k_i - q)$, must also be put in the integrals corresponding to these diagrams. The different diagrams involve permutations of the boson momenta, which concern, however, independent sets on line 1 and line 2, respectively. Thus, according to our numbering of photon lines, the set of internal momenta appearing on line 1 is $(k_1, k_2 + k_3, \dots, k_{2n} + k_{2n+1})$, while the set of internal momenta appearing on line 2 is $(k_1 + k_2, k_3 + k_4, \dots, k_{2n-1} + k_{2n}, k_{2n+1})$. Furthermore, each set satisfies the equation:

$$\sum_{i=1}^{2n+1} k_{iL} = 0. \quad (8.1)$$

Since the integrals we are now calculating are of the regular type, we can, from the start, neglect the transverse momenta k_i^T in the boson propagators (however, this is not compulsory for the following calculations). Application of the permutation formulas (7.3) and (7.5)-(7.7) for the longitudinal momenta independently on lines 1 and 2 yields for the boson propagators the result (notice that formulas (7.5)-(7.7) applied on line 1 with propagators of the type $(-2p_1 \cdot k'_i + i\epsilon)$ still produce $(-2i\pi)$ -factors):

$$I_{2n+1}^{(1,2)} = (-2i\pi)^{2n} \frac{1}{(2p_{1L})^n} \frac{1}{(2p_{2L})^n} \prod_{i=1}^{2n} \delta(k_{iL}), \quad (8.2)$$

and the corresponding potential becomes:

$$\begin{aligned} V^{(2n+1)} &= \frac{i}{2P_L} (-2im_1g)(-2im_2g)(-2ig^2)^{2n} i^{2n} i^{2n+1} (-i)^{2n} \frac{1}{(2p_{1L})^n} \frac{1}{(2p_{2L})^n} \\ &\times \int \left(\prod_{i=1}^{2n+1} \frac{d^3 k_i^T}{(2\pi)^3} \frac{1}{(k_i^{T2} + i\epsilon)} \right) (2\pi)^3 \delta^3 \left(\sum_{j=1}^{2n+1} k_j^T - q^T \right), \end{aligned} \quad (8.3)$$

yielding in x -space:

$$V^{(2n+1)} = -\frac{4m_1m_2}{2P_L} \frac{1}{(p_{1L}p_{2L})^n} \left(\frac{\alpha}{r}\right)^{2n+1}. \quad (8.4)$$

Notice that this formula is also applicable to the case of one-photon exchange [Eq. (5.13)], for which it provides the exact result.

We next consider the diagrams of Fig. 9, with $(2n+2)$ exchanged photons ($n \geq 0$). Here, we have to distinguish between the two types of diagram, according to whether the photon lines begin and end on line 1 (Fig. 9a) or on line 2 (Fig. 9b). We designate by $V^{(1)(2n+2)}$ and $V^{(2)(2n+2)}$ the corresponding potentials, respectively. Let us, for definiteness, consider the diagrams of the first type (Fig. 9a). The number of such diagrams is $\frac{1}{2}(n+2)!(n+1)!$, where the factor $1/2$ comes from the existing symmetry with respect to fixed positions of starting and ending photon lines. The remaining part of the calculation parallels that of $(2n+1)$ exchanged photons. We have on line 1 the set of $(n+2)$ independent internal momenta $(k_1, k_2+k_3, \dots, k_{2n}+k_{2n+1}, k_{2n+2})$ and on line 2 the set of $(n+1)$ independent momenta $(k_1+k_2, k_3+k_4, \dots, k_{2n+1}+k_{2n+2})$, each satisfying Eq. (8.1) (with $2n+1$ replaced by $2n+2$). Since the permutational procedure necessitates $(n+2)!$ permutations on line 1 and $(n+1)!$ permutations on line 2, we have to put the combinatorial factor $1/2$ in front of the final result. We find:

$$\begin{aligned} V^{(1)(2n+2)} &= \frac{i}{2P_L} (-2im_1g)^2 (-2ig^2)^{2n+1} i^{2n+1} i^{2n+2} (-i)^{2n+1} \frac{1}{2} \frac{1}{(2p_{1L})^{n+1}} \frac{1}{(2p_{2L})^n} \\ &\times \int \left(\prod_{i=1}^{2n+2} \frac{d^3k_i^T}{(2\pi)^3 (k_i^{T2} + i\epsilon)} \right) (2\pi)^3 \delta^3 \left(\sum_{j=1}^{2n+2} k_j^T - q^T \right), \end{aligned} \quad (8.5)$$

yielding in x -space:

$$V^{(1)(2n+2)} = \frac{4m_1^2 p_{2L}}{4P_L} \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}}r} \right)^{2n+2}. \quad (8.6)$$

$V^{(2)(2n+2)}$ is obtained from Eq. (8.6) with the exchanges $1 \leftrightarrow 2$.

Collecting all the contributions calculated above, we obtain for the total potential:

$$\begin{aligned} V &= -\frac{2m_1m_2}{P_L} \frac{\alpha}{r} \sum_{n=0}^{\infty} \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}}r} \right)^{2n} \\ &+ \frac{(m_1^2 p_{2L} + m_2^2 p_{1L})}{P_L} \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}}r} \right)^2 \sum_{n=0}^{\infty} \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}}r} \right)^{2n}, \end{aligned} \quad (8.7)$$

which on summation becomes:

$$V = -\frac{2m_1m_2}{P_L} \frac{\alpha}{r} + \left(\frac{m_1^2 p_{2L} + m_2^2 p_{1L}}{P_L} - \frac{2m_1m_2}{P_L} \frac{\alpha}{r} \right) \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}r}} \right)^2 / \left(1 - \left(\frac{\alpha}{\sqrt{p_{1L}p_{2L}r}} \right)^2 \right). \quad (8.8)$$

This expression can be further simplified. We used for the evaluation of higher order diagrams approximations where we neglected (in the regular expressions) the transverse momenta of the external particles (as they contribute to nonleading terms). This is equivalent to using the static limit in these diagrams. Therefore we are entitled to use the equations of motion in the static limit in the corresponding expressions to eliminate energy factors in terms of the masses and energy independent potentials. Using the equations of motion (2.1a)-(2.1b) in the static limit ($p^{T2} = 0$) with V given by Eq. (8.8), we can, after a few algebraic manipulations, reexpress the second piece of V (which involves higher order terms only) in terms of simpler quantities. The final result for V is:

$$V = -\frac{2m_1m_2}{P_L} \frac{\alpha}{r} + \frac{\alpha^2}{r^2}. \quad (8.9)$$

Potential V above has the same structure as the scalar potential proposed by Crater and Van Alstine [27] on the basis of an extension of the minimal substitution rules introduced by Todorov [10], who identified the two-particle motion in the c.m. frame to that of a fictitious particle with reduced mass and energy defined, respectively, as:

$$m_W = \frac{m_1m_2}{P_L}, \quad E = (m_W^2 + b_0^2)^{1/2} = \frac{1}{2P_L}(P^2 - m_1^2 - m_2^2), \quad (8.10)$$

where b_0^2 [Eq. (3.23)] is the mass-shell invariant relative momentum squared, corresponding to the c.m. total energy $\sqrt{s} = P_L$. It can be checked, after rewriting the Klein–Gordon operator H_0 [Eq. (2.11)] in the form

$$H_0 = E^2 - m_W^2 + p^{T2}, \quad (8.11)$$

that potential (8.9) is generated in the wave equations (2.1a)-(2.1b) by the substitution:

$$m_W^2 \rightarrow (m_W + S)^2, \quad (8.12)$$

with $S = -\alpha/r$.

9 The potential in the fermionic case

We turn in this section to the calculation of the scalar and timelike vector potentials relative to fermion-antifermion systems. The fermion propagators are considered in approximations (7.15). According to the analysis, at the end of Sec. 7, of n -photon exchange diagrams, the only surviving terms, at leading order, are those containing in the numerator products of $(n - 1)$ independent combinations of the longitudinal photon momenta k_{iL} . These arise from pairs of fermion and antifermion propagators containing in their numerator one k_{iL} . The permutational procedure already used in other instances produces δ -functions which allow each k_{iL} to cancel the denominator of the corresponding fermion propagator that did not participate in the permutational operation. Hence, there remains at the end the convolution of n photon propagators. The combinatorial analysis is much simplified if we use these cancellations (with the appropriate coefficients) prior to the permutational procedure.

In a diagram with n exchanged photons (Fig. 4) we have several possibilities to choose the $(n - 1)$ independent combinations of k_{iL} 's. Considering for definiteness line 1, we can keep the $(n - 1)$ k_{iL} factors in the numerators of the propagators; then we have to remove the $(n - 1)$ k_{iL} factors from the numerators of the propagators of line 2. We can also keep $(n - 2)$ k_{iL} factors on line 1 and keep one k_{iL} on line 2, the choice of the latter being completely fixed by the complementarity condition to the $(n - 2)$ k_{iL} 's of line 1: the set of the $(n - 1)$ k_{iL} 's globally kept must always form an independent set of vectors. In general, we have the possibility of keeping p k_{iL} 's on line 1 and the complementary $(n - 1 - p)$ k_{iL} 's on line 2.

Let then p be the number of k_{iL} 's kept on line 1. This choice can be made in $\binom{n-1}{p}$ different ways; however, no freedom is left for the choice of the set of $(n - 1 - p)$ complementary k_{iL} 's on line 2. During the permutational procedure, we need $(p + 1)!$ permutations to obtain p δ -functions on line 2 from the propagators not having k_{iL} 's in their numerator; similarly, we need $(n - p)!$ permutations to produce $(n - p - 1)$ δ -functions on line 1. Taking into account the total number of diagrams with n exchanged photons, which is $n!$, we obtain the combinatorial factor relative to the above permutational procedure:

$$C_n = \sum_{p=0}^{n-1} \binom{n-1}{p} \frac{n!}{(p+1)!(n-p)!} = F(-(n-1), -n; 2; 1) = \frac{(2n)!}{(n+1)!n!}, \quad (9.1)$$

where $F(a, b; c; z)$ is the hypergeometric function [48], the value of which, for $z = 1$, is

known in terms of Γ -functions.

After making the approximations $(\gamma_{1L}p_{1L} + m_1) \simeq 2p_{1L}$ and $(-\gamma_{2L}p_{2L} + m_2) \simeq 2p_{2L}$, the integrations on the k_{iL} 's produce the global factor $((-2i\pi)/(2P_L))^{n-1}$. The corresponding potential becomes, in the scalar interaction case:

$$\begin{aligned} \tilde{V}_S^{(n)} &= \frac{i}{2P_L} (-ig)^{2n} i^{2(n-1)} i^n \left(\frac{-i}{2P_L} \right)^{n-1} \frac{(2n)!}{(n+1)!n!} \\ &\times \int \left[\prod_{i=1}^n \frac{d^3 k_i^T}{(2\pi)^3} \frac{1}{(k_i^{T2} + i\epsilon)} \right] (2\pi)^3 \delta^3 \left(\sum_{j=1}^n k_j^T - q \right), \end{aligned} \quad (9.2)$$

yielding in x -space:

$$\tilde{V}_S^{(n)} = \frac{(2n)!}{(n+1)!n!} \left(-\frac{\alpha}{2P_L r} \right)^n. \quad (9.3)$$

This expression is also valid for the one-photon exchange case [Eq. (6.7)], for which it provides the exact result. The two-photon exchange result [Eq. (6.15)] is also reproduced.

The total scalar potential becomes:

$$\tilde{V}_S = \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n+1)!n!} \left(\frac{\alpha}{2P_L r} \right)^n = \frac{1 - \sqrt{1 + \frac{2\alpha}{P_L r}}}{1 + \sqrt{1 + \frac{2\alpha}{P_L r}}}. \quad (9.4)$$

It satisfies the inequality (2.20), and hence parametrizations (2.22) and (2.26) can be used. One obtains:

$$\begin{aligned} \tilde{V}_S &= \tanh V_1, \\ V_1 &= -\frac{1}{4} \ln \left(1 + \frac{2\alpha}{P_L r} \right). \end{aligned} \quad (9.5)$$

The timelike vector potential is obtained from the scalar one, by multiplying, for an n -photon exchange diagram, $\tilde{V}_S^{(n)}$ by $(-1)^n (\gamma_{1L} \gamma_{2L})^n$, the factor -1 coming from the vector photon propagator and the factor $\gamma_{1L} \gamma_{2L}$ from the vertices. Replacing $(n-1)$ factors $\gamma_{1L} \gamma_{2L}$ by their eigenvalue -1 at leading order, we end up with the global factor $-\gamma_{1L} \gamma_{2L}$ with respect to $\tilde{V}_S^{(n)}$. We thus find:

$$\tilde{V}_V^{(n)} = -\frac{(2n)!}{(n+1)!n!} \left(-\frac{\alpha}{2P_L r} \right)^n \gamma_{1L} \gamma_{2L}, \quad (9.6)$$

leading for the total potential to the expression:

$$\tilde{V}_V = -\gamma_{1L} \gamma_{2L} \left(\frac{1 - \sqrt{1 + \frac{2\alpha}{P_L r}}}{1 + \sqrt{1 + \frac{2\alpha}{P_L r}}} \right). \quad (9.7)$$

In terms of parametrizations (2.22) and (2.26) we have:

$$\begin{aligned}\tilde{V}_V &= \tanh(\gamma_{1L}\gamma_{2L}V_2) , \\ V_2 &= \frac{1}{4} \ln \left(1 + \frac{2\alpha}{P_L r} \right) .\end{aligned}\tag{9.8}$$

If we adopt for the electromagnetic potential the hypothesis that in the Feynman gauge relations (6.11), obtained in lowest order, remain also valid in higher orders, we can reconstitute the entire potential:

$$\begin{aligned}V_2 = U_4 &= \frac{1}{4} \ln \left(1 + \frac{2\alpha}{P_L r} \right) , \quad T_4 = 0 , \\ \tilde{V}_V &= \tanh(\gamma_1 \cdot \gamma_2 V_2) .\end{aligned}\tag{9.9}$$

This potential is the fermionic generalization of Todorov's potential [10], obtained in two spin-0 particle systems from Eqs. (8.10) and (8.11) with the substitution rule

$$E^2 \rightarrow (E + V_0)^2 ,\tag{9.10}$$

where $V_0 = \alpha/r$. Indeed, if we go back to Eq. (2.28), which essentially represents the squared two-body Dirac equation, and replace there the potentials V_2 , U_4 and T_4 by their expressions (9.9), we find that in the classical part of this equation the electromagnetic interaction is introduced with the substitution rule (9.10).

As to the scalar potential, it does not correspond in its form (9.4)-(9.5) to the substitution rule (8.12), but can be made compatible with it if in the higher order terms the classical static equations of motion are used, as in the bosonic case [Eq. (8.9)]. To this end, let us replace V_1 by its expression (9.5) in Eq. (2.28) and retain in the latter the classical static (and spin independent) part ($p^T = 0$):

$$\frac{P^2}{4} + \frac{(m_1^2 - m_2^2)^2}{4P^2} - \frac{M^2}{4} e^{4V_1} - \frac{(m_1^2 - m_2^2)^2}{4M^2} e^{-4V_1} \approx 0 ,\tag{9.11}$$

from which we deduce the relation

$$P_L \approx M e^{2V_1} ,\tag{9.12}$$

to be used in higher order terms. Equation (9.11) can also be rewritten in the form:

$$\begin{aligned}\frac{P^2}{4} + \frac{(m_1^2 - m_2^2)^2}{4P^2} - \frac{1}{2}(m_1^2 + m_2^2) - m_1 m_2 (1 - e^{-4V_1}) \\ - \frac{M^2}{4} e^{4V_1} (1 - e^{-4V_1})^2 \approx 0 .\end{aligned}\tag{9.13}$$

Using in the last term, which concerns higher order terms than the first, Eq. (9.2) for the factor $M^2 e^{4V_1}$ we find:

$$\frac{P^2}{4} + \frac{(m_1^2 - m_2^2)^2}{4P^2} - \frac{1}{2}(m_1^2 + m_2^2) - \frac{2m_1 m_2}{P_L} S + S^2 \approx 0, \quad (9.14)$$

where S is defined as:

$$S = -\frac{\alpha}{r}. \quad (9.15)$$

The interaction dependent part in Eq. (9.14) has the same form as in the bosonic case [Eq. (8.9)], after a similar procedure was used there, and corresponds to the substitution rule (8.12).

This agreement can be considered as a consistency check for our approximation scheme, applied to bosons and fermions with different diagrammatic structures. As one would naturally expect, the classical parts of the corresponding effective interactions do coincide.

Adopting Eq. (9.14) as the final form of the classical part of the effective interaction, one can reconstruct the new expression of V_1 through the identification of the classical static part of Eq. (2.28) with Eq. (9.14). One finds:

$$V_1 = \frac{1}{2} \ln \left(\frac{1}{M} \left[\left(m_1^2 + \frac{2m_1 m_2}{P_L} S + S^2 \right)^{\frac{1}{2}} + \left(m_2^2 + \frac{2m_1 m_2}{P_L} S + S^2 \right)^{\frac{1}{2}} \right] \right), \quad (9.16)$$

with S defined in Eq. (9.15).

The results obtained thus far can be generalized to include the case of a mixture of scalar and vector interactions. The calculations can be repeated as above, with the difference that every scalar photon propagator (of the scalar interaction case) is now replaced by an effective propagator that is the sum of the scalar and vector propagators (including the couplings at the vertices):

$$g^2 D_S \rightarrow g^2 D_S + e^2 \gamma_{1L} \gamma_{2L} D_V. \quad (9.17)$$

It is this sum which is factorized in the Feynman diagram integrals, while the fermion propagator part remains unaffected. Concerning the determination of the tensor nature of the interference terms, the approximations utilized in the fermionic case (in particular, the replacements in several places of the matrix products $\gamma_{1L} \gamma_{2L}$ by -1) do not leave enough predictivity about this question. It turns out, however, that the analysis of the same problem in the bosonic case is more predictive and suggestive of the representation

of the interference terms by effective scalar interactions. We shall adopt this property also for the fermionic case.

When the summation is done with the effective propagator (9.17), the pure vector potential, which has the form (9.7)-(9.8), can be isolated from the total potential and $\gamma_{1L}\gamma_{2L}$ replaced in the rest by -1 . The result is:

$$\begin{aligned}\tilde{V} &= \tanh(V_1 + \gamma_{1L}\gamma_{2L}V_2) , \\ V_2 &= \frac{1}{4} \ln(1 + \frac{2}{P_L}V_0) , \quad V_0 = \frac{\alpha}{r} , \\ V_1 &= -\frac{1}{4} \ln \left(1 - \frac{\frac{2}{P_L}S}{1 + \frac{2}{P_L}V_0} \right) , \quad S = -\frac{\alpha'}{r} , \quad \alpha' = \frac{g^2}{4\pi} .\end{aligned}\tag{9.18}$$

Including then the spacelike potentials U_4 and T_4 with the hypothesis (9.9), we obtain:

$$\begin{aligned}\tilde{V} &= \tanh(V_1 + \gamma_{1L}\gamma_{2L}V_2 + \gamma_1^T \cdot \gamma_2^T U_4) , \\ U_4 = V_2 &= \frac{1}{4} \ln \left(1 + \frac{2}{P_L}V_0 \right) .\end{aligned}\tag{9.19}$$

We again can transform the scalar potential with the aid of the classical static equations of motion, as in Eqs. (9.11)-(9.16). We end up with the following expression of V_1 :

$$\begin{aligned}V_1 &= \frac{1}{2} \ln \left(\frac{1}{M\sqrt{1 + \frac{2}{P_L}V_0}} \left[\left(m_1^2(1 + \frac{2}{P_L}V_0) + \frac{2m_1m_2}{P_L}S + S^2 \right)^{\frac{1}{2}} \right. \right. \\ &\quad \left. \left. + \left(m_2^2(1 + \frac{2}{P_L}V_0) + \frac{2m_1m_2}{P_L}S + S^2 \right)^{\frac{1}{2}} \right] \right) , \quad V_0 = \frac{\alpha}{r} , \quad S = -\frac{\alpha'}{r} ,\end{aligned}\tag{9.20}$$

with V_2 and U_4 given by Eqs. (9.19). The classical parts of the above potentials satisfy Todorov's minimal substitution rules [10], defined in Eqs. (8.10)-(8.11), (8.12) and (9.10).

10 Reconstitution of the Bethe–Salpeter wave function

In this section we shall reconstitute, in the fermionic case, the Bethe-Salpeter wave function in the framework of the local approximation that we used throughout this work. The exact reconstitution formula is given by Eq. (3.17), where in the left argument of the first T the constraint (2.8) is not used. A first approximation consists in replacing the factor $T(1 - \tilde{g}T)^{-1}$ by potential \tilde{V} [Eq. (3.19)] in which, however, p_L is no longer submitted to the constraint (2.8). This is easily done by leaving, in the expression $\tilde{V}(t, P_L)$, calculated in the local approximation, the part q_L^2 of $t(= q_L^2 + q^{T2})$ as a free variable, not fixed at its zero value. More precisely, $q_L = p_L - p'_L = p_L - (m_1^2 - m_2^2)/(2P_L)$, where p'_L acts on the right on $\tilde{\psi}$ and hence is submitted to the constraint (2.8). We therefore obtain:

$$\phi(p) = S_1(p_1)S_2(-p_2)\frac{2P_L}{i} \int \frac{d^3p'^T}{(2\pi)^3} \tilde{V}\left((p_L - \frac{(m_1^2 - m_2^2)}{2P_L})^2 + (p^T - p'^T)^2, P_L\right) \tilde{\psi}(p'^T) . \quad (10.1)$$

A further approximation can be imposed by neglecting altogether the q_L^2 dependence of \tilde{V} in Eq. (10.1). In this case the wave equation (3.20), together with Eq. (6.5), can be used and one obtains:

$$\phi = -S_1S_2\frac{2P_L}{i} \left[S_1S_2H_0 \right]_{C(p)}^{-1} \tilde{\psi} . \quad (10.2)$$

The relative longitudinal momentum is now contained only in the two propagators S_1 and S_2 . Formulas (10.1) or (10.2) can be used in the calculations of transition amplitudes, form factors and decay coupling constants.

Of particular interest is the expression of the Bethe–Salpeter wave function at the zero value of the relative longitudinal coordinate [it enters in the calculation of decay coupling constants]:

$$\phi(x_L = 0) = \int \frac{dp_L}{2\pi} \phi(p_L) . \quad (10.3)$$

Adopting approximation (10.2), integration on p_L concerns only the propagators S_1 and S_2 . One finds:

$$\begin{aligned} \int \frac{dp_L}{2\pi} S_1(p_1)S(-p_2) &= \frac{i}{4P_L H_0} \left\{ (\gamma_1 \cdot p_1 + m_1)(-\gamma_2 \cdot p_2 + m_2) \left(\frac{p_{1L}}{E_1} + \frac{p_{2L}}{E_2} \right) \right. \\ &\quad \left. - \left(\frac{1}{E_1} - \frac{1}{E_2} \right) H_0 \left[\gamma_{2L}(\gamma_1 \cdot p_1 + m_1) + \gamma_{1L}(-\gamma_2 \cdot p_2 + m_2) \right] \right\} \end{aligned}$$

$$+\gamma_{1L}\gamma_{2L}H_0\left(\frac{p_{1L}}{E_1}+\frac{p_{2L}}{E_2}\right)\Big\}_{C(p)} , \quad (10.4)$$

where E_1 and E_2 are defined as

$$E_1 = \sqrt{m_1^2 - p^{T2}} , \quad E_2 = \sqrt{m_2^2 - p^{T2}} , \quad (10.5)$$

and constraint (2.8) is applied in the right-hand side [p_{1L} and p_{2L} are given by Eqs. (2.9)].

In the framework of local potential approximation, we can replace the nonlocal operators E_1 and E_2 by the eigenvalues p_{1L} and p_{2L} , respectively [Eqs. (2.9)]. Equations (10.2)-(10.4) then yield:

$$\begin{aligned} \phi(x_L = 0, x^T) = & \left\{ 1 - \frac{1}{2} \left(\frac{1}{p_{1L}} - \frac{1}{p_{2L}} \right) \left[\gamma_{2L}(-\gamma_2 \cdot p_2 - m_2) + \gamma_{1L}(\gamma_1 \cdot p_1 - m_1) \right] \right. \\ & \left. + \gamma_{1L}\gamma_{2L} \frac{1}{H_0} (\gamma_1 \cdot p_1 - m_1)(-\gamma_2 \cdot p_2 - m_2) \right\} \tilde{\psi}(x^T) . \end{aligned} \quad (10.6)$$

After the use of the wave equations (2.13a)-(2.13b), Eq. (10.6) becomes:

$$\begin{aligned} \phi(x_L = 0, x^T) = & \left\{ 1 + \gamma_{1L}\gamma_{2L}\tilde{V} - \frac{1}{2} \left(\frac{1}{p_{1L}} - \frac{1}{p_{2L}} \right) \left[\gamma_{2L}(\gamma_1 \cdot p_1 + m_1) \right. \right. \\ & \left. \left. + \gamma_{1L}(-\gamma_2 \cdot p_2 + m_2) \right] \tilde{V} \right\} \tilde{\psi}(x^T) . \end{aligned} \quad (10.7)$$

Finally, using parametrization (2.22) and the wave function transformation (2.23) one obtains:

$$\begin{aligned} \phi(x_L = 0, x^T) = & \left\{ e^{\gamma_{1L}\gamma_{2L}V} - \frac{1}{2} \left(\frac{1}{p_{1L}} - \frac{1}{p_{2L}} \right) \left[\gamma_{2L}(\gamma_1 \cdot p_1 + m_1) \right. \right. \\ & \left. \left. + \gamma_{1L}(-\gamma_2 \cdot p_2 + m_2) \right] \sinh V \right\} \psi(x^T) , \end{aligned} \quad (10.8)$$

where ψ is normalized according to Eq. (2.24) and V is of the type (2.26) and is a function of x^T and P_L .

11 Conclusion

Constraint theory, applied to two-particle systems, provides a natural basis for a manifestly covariant three-dimensional reduction of the Bethe–Salpeter equation. The two-body potential is related, as in the quasipotential approach, to the off-mass shell scattering amplitude by means of a Lippmann–Schwinger type equation and is calculable in terms of Feynman diagrams. Perturbation theory is reorganized with the presence of “constraint diagrams” that appear in the course of the three-dimensional reduction process.

The two-photon exchange diagrams (for scalar interactions or vector interactions considered in the Feynman gauge) are globally free of spurious infrared singularities and yield at leading order local potentials proportional to $(\alpha/r)^2$, where r is the three-dimensional relative distance in the c.m. frame; these, together with the one-photon exchange contribution, produce the correct bound state spectra to order α^4 . The n -photon exchange diagram contributions were evaluated in an approximation scheme that is a variant of the eikonal approximation adapted to the bound state problem; the latter produce at leading order local potentials proportional to $(\alpha/r)^n$. The series of leading order potentials were summed, producing total potentials that are functions of r and of the c.m. total energy. The potentials calculated for vector and scalar interactions are equivalent, for their classical parts, to those obtained by Todorov [10] and Crater and Van Alstine [27] on the basis of minimal substitution rules.

In summing the series of leading order potentials, ambiguities might arise from possible incorporation or omission of nonleading terms. A crucial restriction imposed on such ambiguities comes from an inequality [(2.20)] that should satisfy the fermionic potential in order to ensure the positivity of the norm of the wave function. The potentials obtained in this work do satisfy the above inequality and hence do not lead to spurious violations of the positivity of the norm.

Another question that arises concerns the utility of the potential formalism itself for the evaluation of the bound state spectrum: would not the scattering amplitude, calculated in the eikonal approximation, provide the bound state spectrum through the determination of the positions of its poles? The accuracy of the evaluation of the potential in the present work goes beyond that of the conventional eikonal approximation. For the bosonic case for instance, the latter sums diagrams of the type of Fig. 4. From the potential theory viewpoint, the net effect of the contribution of such diagrams is equivalent to the one-photon

exchange contribution (Fig. 1). It is the summation of the chains of seagull diagrams of the types of Figs. 3, 8 and 9, not taken into account in conventional eikonal approximation, that provides the genuine multiphoton-exchange contributions to the potential. Similarly, in the fermionic case, the contributions isolated in the present work for the evaluation of the potential are not considered in conventional eikonal approximation; furthermore, the latter would produce a spin independent bound state spectrum. On the other hand, the potential evaluated above is reinjected into relativistic wave equations that take into account more accurately spin dependent effects. We conclude that potential formalism is necessary for a detailed probe of the relativistic bound state spectrum.

The generality of the summation method, in which the photon propagators, except for the counting rules, do not play any active role, leaves open the possibility of extending the above calculations to other types of interaction, like those corresponding to the exchanges of massive particles or those corresponding to effective confining interactions. Another domain of application might concern the study of the gauge dependence problem of the electromagnetic two-body potential.

The fact that the constraint theory wave equations can be reduced to a single Pauli-Schrödinger type equation [29], provides the complementary basis for their simple applicability to relativistic bound state problems.

A Calculation of the scalar integrals

In this appendix we calculate the scalar integrals defined in Eqs. (5.15)-(5.17). They have already been evaluated in the literature [42, 43], but generally on the mass shell, with a small mass given to the photon. Since we are using an off-mass shell formalism, we present here some details of the calculations. Definitions of variables and approximations were introduced at the beginning of Sec. 5, while the counting rules of the orders of magnitude were explained in Sec. 4; we recall that these are referred to x -space. We use for the calculation of the integrals the Feynman parametrization method.

We first consider the integral J . After an integration on one of the parameters, we find:

$$J = i\pi^2 \int_0^1 \int_0^1 \frac{d\gamma d\sigma}{[\gamma\lambda^2 - (1-\gamma)t][\lambda^2 + \gamma[\sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)s]]} . \quad (\text{A.1})$$

We next integrate with respect to σ ; the result is an arctan function, from which we isolate the dominant part by expressing it with respect to the inverse of its argument. In the nondominant part we make the change of variable $x = \sqrt{\frac{\lambda^2}{\gamma s} - \frac{b^2}{s}}$. We find:

$$J = j + \tilde{J} , \quad (\text{A.2})$$

with

$$j = \frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma[\gamma(\lambda^2 + t) - t]\sqrt{\frac{\lambda^2}{\gamma s} - \frac{b^2}{s}}} = -\frac{i\pi^3}{\sqrt{s}} \frac{1}{\sqrt{t(-b_0^2 t + (\lambda^2)^2)}} \ln \left(\frac{1 - \sqrt{1 - \frac{(\lambda^2)^2}{b_0^2 t}}}{1 + \sqrt{1 - \frac{(\lambda^2)^2}{b_0^2 t}}} \right) \quad (\text{A.3})$$

and

$$\tilde{J} = \frac{2i\pi^2}{st} \int_{\sqrt{-b_0^2/s}}^\infty \frac{dx}{[x^2 - ((\lambda^2)^2 - b_0^2 t)/(st)]} \left(\arctan\left(\frac{x}{\beta_1}\right) + \arctan\left(\frac{x}{\beta_2}\right) \right) , \quad (\text{A.4})$$

where

$$\beta_1 = \frac{p_{1L}}{P_L} , \quad \beta_2 = \frac{p_{2L}}{P_L} . \quad (\text{A.5})$$

The integration domain of x in Eq. (A.4) is separated into two parts by a point lying in the region of the x 's of the order of $\alpha^{1/2}$. Appropriate approximations of the integrand

in each of the domains lead to the following expression of \tilde{J} :

$$\begin{aligned} \tilde{J} = & \frac{2i\pi^2}{st} \left[-\frac{1}{2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \ln \left(-\frac{(\lambda^2)^2}{st} \right) + \left(\frac{1}{\beta_1} \ln \beta_1 + \frac{1}{\beta_2} \ln \beta_2 \right) \right. \\ & \left. + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right] + O(\alpha^3 \ln \alpha^{-1}) . \end{aligned} \quad (\text{A.6})$$

$J(1, -2')$ is obtained from Eq. (A.1) by the replacement of s by u [we are working in the approximations $p_2'^2 = p_2^2$ and $\lambda'^2 = \lambda^2$]. One first integrates with respect to γ :

$$\begin{aligned} J(1, -2') = & -\frac{i\pi^2}{t} \int_0^1 \frac{d\sigma}{\left[\sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)u + \lambda^2(\lambda^2 + t)/t \right]} \\ & \times \ln \left(-\frac{[\sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)u + \lambda^2]t}{(\lambda^2)^2} \right) . \end{aligned} \quad (\text{A.7})$$

In the bound state domain we are considering, we have the following approximate expressions for s and u :

$$\begin{aligned} s = & \left(\sqrt{p_1^2 + b^2} + \sqrt{p_2^2 + b^2} \right)^2 \simeq \left(\sqrt{p_1^2} + \sqrt{p_2^2} \right)^2 \left(1 + \frac{b^2}{\sqrt{p_1^2 p_2^2}} \right) , \\ u \simeq & \left(\sqrt{p_1^2} - \sqrt{p_2^2} \right)^2 - t - b^2 \frac{(\sqrt{p_1^2} + \sqrt{p_2^2})^2}{\sqrt{p_1^2 p_2^2}} , \end{aligned} \quad (\text{A.8})$$

which allow us to write:

$$\begin{aligned} \sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)u \simeq & \left(\sqrt{p_2^2} + \sigma(\sqrt{p_1^2} - \sqrt{p_2^2}) \right)^2 \\ & + \sigma(1-\sigma) \left[t + b^2 \frac{(\sqrt{p_1^2} + \sqrt{p_2^2})^2}{\sqrt{p_1^2 p_2^2}} \right] ; \end{aligned} \quad (\text{A.9})$$

the second term in the right-hand side is negligible in front of the first and this leads to the following expression of $J(1, -2')$:

$$\begin{aligned} J(1, -2') = & -\frac{2i\pi^2}{st} \left[-\frac{1}{2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \ln \left(-\frac{(\lambda^2)^2}{st} \right) \right. \\ & \left. - \frac{1}{\beta_1(\beta_1 - \beta_2)} \ln \beta_1 + \frac{1}{\beta_2(\beta_1 - \beta_2)} \ln \beta_2 + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right] + O(\alpha^3 \ln \alpha^{-1}) . \end{aligned} \quad (\text{A.10})$$

The constraint integral J^C , which is three-dimensional, is also calculated with the Feynman parametrization method. After one integration one finds:

$$J^C = -\frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma[\gamma\lambda^2 - (1-\gamma)t]\sqrt{\frac{\lambda^2}{\gamma s} - \frac{b^2}{s}}} = -j . \quad (\text{A.11})$$

The structure of the F -integrals is very similar to that of the J -integrals (A.1) and the same types of approximations are applied. We obtain:

$$F = f + \tilde{F} , \quad (\text{A.12})$$

$$f = -\frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma \sqrt{\frac{\lambda^2}{\gamma s} - \frac{b^2}{s}}} = -\frac{2i\pi^3}{s} \sqrt{\frac{s}{b^2}} \arctan \sqrt{\frac{b^2}{b_0^2}} , \quad (\text{A.13})$$

$$\begin{aligned} \tilde{F} &= -\frac{2i\pi^2}{s} \int_{\sqrt{-b_0^2/s}}^{\infty} \frac{dx}{(x^2 + b^2/s)} \left[\arctan\left(\frac{x}{\beta_1}\right) + \arctan\left(\frac{x}{\beta_2}\right) \right] , \\ &= \frac{2i\pi^2}{s} \left[-\frac{1}{2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \ln\left(\frac{\lambda^2}{s}\right) + \left(\frac{1}{\beta_1} \ln \beta_1 + \frac{1}{\beta_2} \ln \beta_2 \right) + \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \right] + O(\alpha^5 \ln \alpha^{-1}) , \end{aligned} \quad (\text{A.14})$$

$$\begin{aligned} F(1, -2') &= -i\pi^2 \int_0^1 \frac{d\sigma}{[\sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)u]} \ln \left(\frac{\sigma p_1^2 + (1-\sigma)p_2^2 - \sigma(1-\sigma)u}{\lambda^2} \right) \\ &= -\frac{2i\pi^2}{s} \left[-\frac{1}{2} \left(\frac{1}{\beta_1} + \frac{1}{\beta_2} \right) \ln\left(\frac{\lambda^2}{s}\right) - \frac{1}{(\beta_1 - \beta_2)} \left(\frac{1}{\beta_1} \ln \beta_1 - \frac{1}{\beta_2} \ln \beta_2 \right) \right] \\ &\quad + O(\alpha^5 \ln \alpha^{-1}) , \end{aligned} \quad (\text{A.15})$$

$$F^C = \frac{i\pi^3}{s} \int_0^1 \frac{d\gamma}{\gamma \sqrt{\frac{\lambda^2}{\gamma s} - \frac{b^2}{s}}} = -f . \quad (\text{A.16})$$

Notice that with the approximations $p_1^2 = p_1'^2$, $p_2^2 = p_2'^2$, $\lambda^2 = \lambda'^2$ we have used, the F -integrals are independent of t and hence

$$H = F , \quad H(1, -2') = F(1, -2') , \quad H^C = F^C . \quad (\text{A.17})$$

The G -integrals can be expressed in terms of Euler's dilogarithm or the Spence function. One finds (without the approximations $p_1^2 = p_1'^2$ and $p_2^2 = p_2'^2$):

$$G^{(1)} = -\frac{i\pi^4}{\sqrt{-2t(p_1^2 + p_1'^2) + t^2 + (p_1^2 - p_1'^2)^2}} + O(\alpha^3 \ln \alpha^{-1}) , \quad (\text{A.18})$$

the dominant part of it being:

$$G^{(1)} \simeq -\frac{i\pi^4}{2p_{1L}\sqrt{-t}} . \quad (\text{A.19})$$

$G^{(2)}$ is obtained from the above equations by the replacements $1 \leftrightarrow 2$.

$G^{(1)}$ does not have a crossed-diagram counterpart, since it depends only on the particle 1 momenta. On the other hand, the constraint integral G^C [Eq. (5.17)] does not

correspond to an explicit diagram in the expansion (4.4), but rather may appear through integrals involving fermions or vector interactions of bosons. One finds:

$$G^C = \frac{2i\pi^3}{2P_L} \int_0^1 \frac{d\gamma}{\sqrt{-\gamma(1-\gamma)t}} = \frac{i\pi^4}{P_L\sqrt{-t}} . \quad (\text{A.20})$$

We emphasize that the instantaneous approximation in the photon propagators (setting there $k_L = 0$) would not produce the results found above. For instance, for the G -integral one would find twice the correct result (A.19). A detailed analysis of the integral shows that the photon propagator poles in k_L contribute with a factor $-1/2$ with respect to the contribution of the boson propagator poles. However, the linearization approximation of the inverse bosonic propagators, as used starting from Sec. 7, makes the instantaneous approximation compatible with the leading order results for the potential.

B Calculation of the vector and tensor integrals

Most of the vector and tensor integrals can be calculated by algebraic decompositions with the aid of the vectors p_1 , p_2 , q , etc., in terms of the scalar integrals calculated in Appendix A, except for those decompositions that might involve infra-red divergences, in which case a direct calculation is necessary. We present here the final results [notations are those of Appendix A]:

$$F_\mu = q_\mu f - \frac{i\pi^2}{s}(\beta_2 p_{1\mu} - \beta_1 p_{2\mu} - q_\mu) \left[-\left(\frac{1}{\beta_1} + \frac{1}{\beta_2}\right) + \pi \sqrt{\frac{s}{b^2}} \left(\frac{\lambda^2}{b^2} \arctan \sqrt{-\frac{b^2}{b_0^2}} - \sqrt{-\frac{b_0^2}{b^2}} \right) \right] + O(\alpha^4 \ln \alpha^{-1})$$

$$\equiv q_\mu f + p_\mu'^T \widetilde{H} , \quad (B.1)$$

$$H_\mu = F_\mu(q=0) = p_\mu^T \widetilde{H} , \quad (B.2)$$

$$F_\mu(1, -2') = -\frac{i\pi^2}{s} \left\{ p_{1\mu}' \frac{1}{\beta_1 \beta_2} + (p_{2\mu}' - p_{1\mu}) \left[-\frac{1}{(\beta_1 - \beta_2)^2} \ln\left(\frac{\beta_1}{\beta_2}\right) + \frac{1}{2\beta_1 \beta_2} + \frac{1}{2\beta_1 \beta_2} \left(\frac{\beta_1 + \beta_2}{\beta_1 - \beta_2} \right) \right] \right\} + O(\alpha^4 \ln \alpha^{-1}) , \quad (B.3)$$

$$H_\mu(1, -2') = \frac{i\pi^2}{s} \left\{ -p_{1\mu} \frac{1}{\beta_1 \beta_2} + (p_{1\mu}' - p_{2\mu}) \left[-\frac{1}{(\beta_1 - \beta_2)^2} \ln\left(\frac{\beta_1}{\beta_2}\right) + \frac{1}{2\beta_1 \beta_2} + \frac{1}{2\beta_1 \beta_2} \left(\frac{\beta_1 + \beta_2}{\beta_1 - \beta_2} \right) \right] \right\} + O(\alpha^4 \ln \alpha^{-1}) , \quad (B.4)$$

$$F_\mu^C = q_\mu F^C + \frac{i\pi^3}{s} p_\mu'^T \sqrt{\frac{s}{b^2}} \left(\frac{\lambda^2}{b^2} \arctan \sqrt{-\frac{b^2}{b_0^2}} - \sqrt{-\frac{b_0^2}{b^2}} \right) + O(\alpha^4 \ln \alpha^{-1})$$

$$\equiv q_\mu F^C + p_\mu'^T \widetilde{H}^C , \quad (B.5)$$

$$H_\mu^C = \frac{i\pi^3}{s} p_\mu^T \sqrt{\frac{s}{b^2}} \left(\frac{\lambda^2}{b^2} \arctan \sqrt{-\frac{b^2}{b_0^2}} - \sqrt{-\frac{b_0^2}{b^2}} \right) + O(\alpha^4 \ln \alpha^{-1})$$

$$= p_\mu^T \widetilde{H}^C , \quad (B.6)$$

$$G_\mu^{(1)} = -\frac{\lambda^2}{(p_1^2 + p_1'^2)} p_{1\mu} G^{(1)} + \left\{ \frac{(\lambda^2 + 2p_1^2)}{2(p_1^2 + p_1'^2)} - \frac{\lambda^2(p_1^2 - p_1'^2)}{2t(p_1^2 + p_1'^2)} + \frac{p_1^2(p_1^2 - p_1'^2)^2}{2t(p_1^2 + p_1'^2)^2} \right\} q_\mu G^{(1)} + O(\alpha^4)$$

$$\simeq \frac{1}{2} q_\mu G^{(1)} , \quad (B.7)$$

$$G_\mu^C = \frac{1}{2} q_\mu G^C . \quad (B.8)$$

$G_\mu^{(2)}$ is obtained from $G_\mu^{(1)}$ by the replacements $p_1 \rightarrow -p_2$, $p'_1 \rightarrow -p'_2$ and $q \rightarrow q$:

$$G_\mu^{(2)} \simeq \frac{1}{2} q_\mu G^{(2)}. \quad (\text{B.9})$$

For the J_μ -integrals one finds:

$$\begin{aligned} J_\mu = & P_\mu \frac{1}{2P^2} (G^{(1)} - G^{(2)}) + q_\mu \frac{1}{2} \left[J + \frac{1}{t} (F - H) \right] \\ & + (p^T + p'^T)_\mu \frac{1}{(p^T + p'^T)^2} \left[-\frac{1}{P_L} (p_{1L} G^{(1)} + p_{2L} G^{(2)}) - (\lambda^2 + \frac{t}{2}) J + \frac{1}{2} (F + H) \right], \end{aligned} \quad (\text{B.10})$$

$$\begin{aligned} J_\mu(1, -2') = & -P_\mu \frac{1}{2P^2} \left[(G^{(1)} + G^{(2)}) + 2(\lambda^2 + \frac{t}{2}) J(1, -2') - (F(1, -2') + H(1, -2')) \right] \\ & + (p^T + p'^T)_\mu \frac{1}{(p^T + p'^T)^2} \left[\frac{1}{P_L} (p_{1L} G^{(1)} - p_{2L} G^{(2)}) \right. \\ & + \frac{(p_{1L} - p_{2L})}{P_L} (\lambda^2 + \frac{t}{2}) J(1, -2') - \frac{(p_{1L} - p_{2L})}{P_L} F(1, -2') \left. \right] \\ & + q_\mu \frac{1}{2} \left[J(1, -2') + \frac{1}{t} (F(1, -2') - H(1, -2')) \right], \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} J_\mu^C = & -(p^T + p'^T)_\mu \frac{1}{(p^T + p'^T)^2} \left[G^C + (\lambda^2 + \frac{t}{2}) J^C - \frac{1}{2} (F^C + H^C) \right] \\ & + q_\mu \frac{1}{2} [J^C + \frac{1}{2} (F^C - H^C)]. \end{aligned} \quad (\text{B.12})$$

In showing Eqs. (6.13) for the transverse components of the J_μ -integrals, one uses in some places the leading order equality (5.11).

For the $J_{\mu\nu}$ -integrals it is preferable to first make a decomposition along the vector integrals already calculated and $g_{\mu\nu}^{TT}$. For the simplicity of presentation, we use certain relations previously obtained (such as $F = H$, etc.) and the leading order equality (5.11), and neglect some nonleading terms. We find:

$$\begin{aligned} J_{\mu\nu} = & (P_\mu q_\nu + P_\nu q_\mu) \frac{1}{4P^2} (G^{(1)} - G^{(2)}) + g_{\mu\nu}^{TT} \frac{1}{2} \left[\frac{1}{P_L} (p_{1L} G^{(1)} + p_{2L} G^{(2)}) + \lambda^2 J \right] \\ & - \left((p^T + p'^T)_\mu q_\nu + (p^T + p'^T)_\nu q_\mu \right) \left\{ \frac{1}{(p^T + p'^T)^2} \left[\frac{1}{2P_L} (p_{1L} G^{(1)} + p_{2L} G^{(2)}) + \frac{1}{2} F \right] \right. \\ & + \frac{1}{4} J \left. \right\} + \left((p^T + p'^T)_\mu P_\nu + (p^T + p'^T)_\nu P_\mu \right) \frac{1}{4P^2} (G^{(1)} - G^{(2)}) \\ & + (p^T + p'^T)_\mu (p^T + p'^T)_\nu \frac{1}{2(p^T + p'^T)^2} \left[-\frac{2}{P_L} (p_{1L} G^{(1)} + p_{2L} G^{(2)}) - (2\lambda^2 + \frac{t}{2}) J \right. \\ & + F + \widetilde{H} \left. \right] + q_\mu q_\nu \frac{1}{2t} \left[(-\lambda^2 + \frac{t}{2}) J + f - \widetilde{H} - \frac{1}{P_L} (p_{1L} G^{(1)} + p_{2L} G^{(2)}) \right], \end{aligned} \quad (\text{B.13})$$

[f and \widetilde{H} are defined in Eq. (B.1),]

$$\begin{aligned}
J_{\mu\nu}(1, -2') &= -P_\mu \frac{1}{P^2} \left[\frac{1}{2} (G_\nu^{(1)} + G_\nu^{(2)}(-2')) + \left(\frac{\lambda^2}{2} + t \right) J_\nu(1, -2') \right] \\
&\quad - \frac{(p_{1L} - p_{2L})}{2P_L} (p^T + p'^T)_\mu J_\nu(1, -2') + \frac{1}{2} q_\mu J_\nu(1, -2') \\
&\quad + g_{\mu\nu}^{TT} \left(1 - \frac{(p_{1L} - p_{2L})^2}{2P_L^2} \right) F(1, -2') + O(\alpha^3) ,
\end{aligned} \tag{B.14}$$

$$\begin{aligned}
J_{\mu\nu}^C &= (p^T + p'^T)_\mu (p^T + p'^T)_\nu \frac{1}{2(p^T + p'^T)^2} \left[\frac{1}{2} (p^T + p'^T)^2 J^C - \lambda^2 J^C - 2G^C \right. \\
&\quad \left. + F^C + \widetilde{H}^C \right] + \left((p^T + p'^T)_\mu q_\nu + (p^T + p'^T)_\nu q_\mu \right) \frac{1}{2(p^T + p'^T)^2} \left[\frac{1}{2} (p^T + p'^T)^2 J^C \right. \\
&\quad \left. - G^C + F^C \right] + q_\mu q_\nu \frac{1}{2t} \left[(-\lambda^2 + \frac{t}{2}) J^C - G^C + F^C - \widetilde{H}^C \right] + g_{\mu\nu}^{TT} \frac{1}{2} (G^C + \lambda^2 J^C) .
\end{aligned} \tag{B.15}$$

[\widetilde{H}^C is defined in Eq. (B.5).]

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Figures

Fig. 1. One-photon exchange diagram.

Fig. 2. Two-photon exchange diagrams; the “constraint diagram” is denoted by a cross.

Fig. 3. Two-photon exchange diagrams contributing to the bosonic case.

Fig. 4. A multiphoton exchange diagram.

Fig. 5. A typical “constraint diagram” where the constraint factor \tilde{g}_0 [Eq. (4.2)] appears twice ($p = 2$).

Fig. 6. Examples of mixtures of elementary diagrams in the bosonic case.

Fig. 7. “Constraint diagrams” associated with diagrams of Fig. 6.

Fig. 8. Typical diagram with an odd number of exchanged scalar photons contributing to the potential in the bosonic case.

Fig. 9. Typical diagrams with an even number of exchanged scalar photons contributing to the potential in the bosonic case.

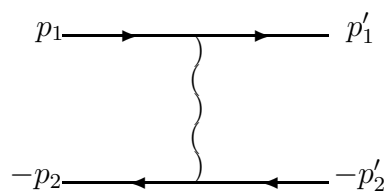


Fig. 1

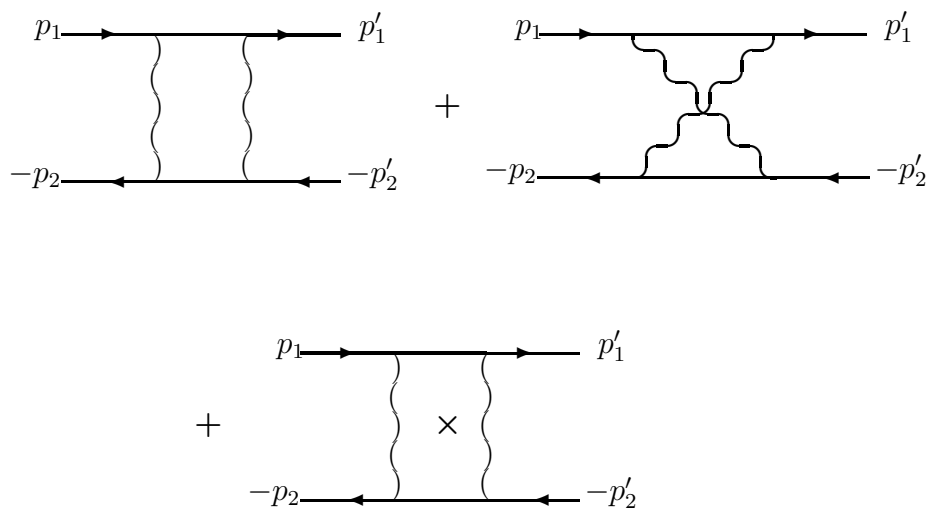


Fig. 2

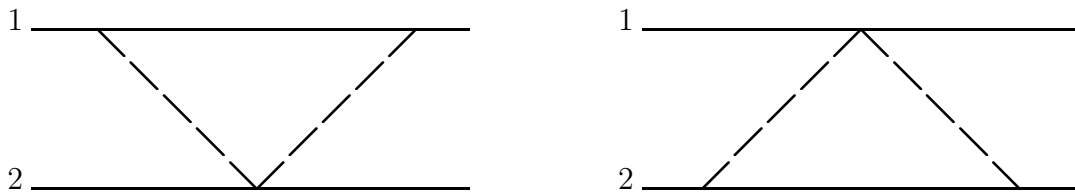


Fig. 3

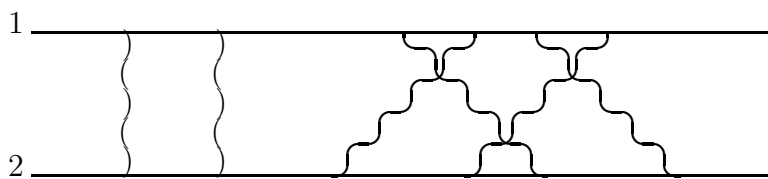


Fig. 4

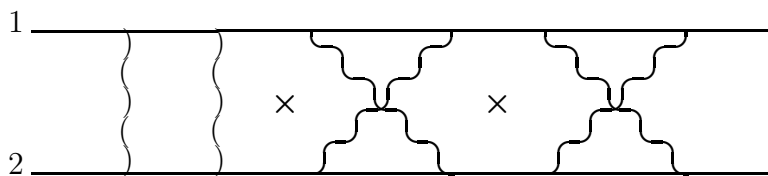


Fig. 5

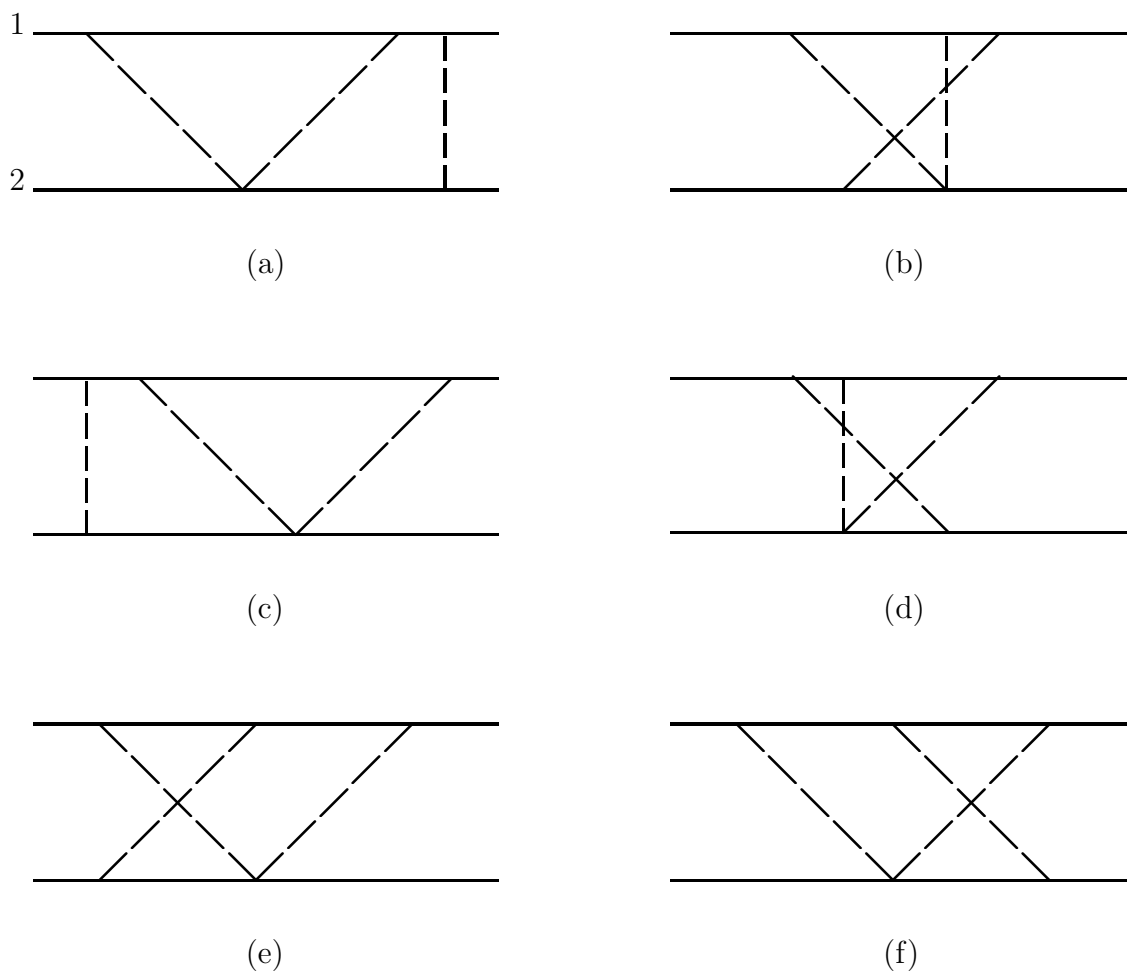


Fig. 6

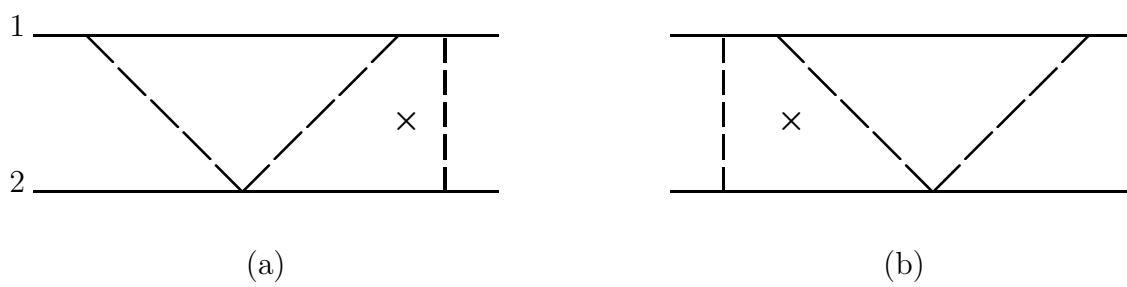


Fig. 7

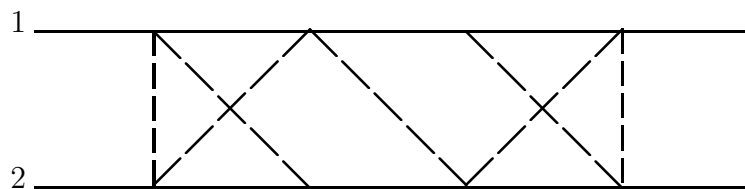
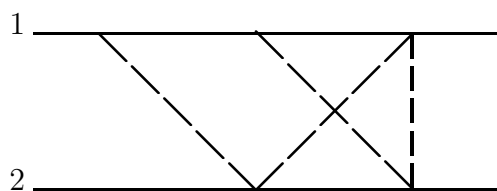
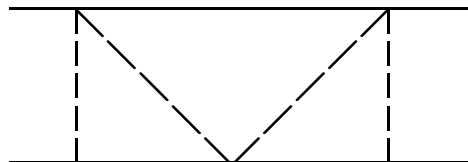


Fig. 8



(a)



(b)

Fig. 9